CLASSROOM NOTES

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MEAN AND ORDINARY CONVERGENCE OF A SEQUENCE OF FUNCTIONS

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In presenting the idea of mean convergence to undergraduates who are not acquainted with the Lebesgue integral the following procedure has been found to be helpful. We point out that mean convergence may apply to cases where ordinary convergence fails, and we also prove that under appropriate conditions ordinary convergence implies mean convergence.

A sequence of functions $f_n(x)$, defined in the finite closed interval (a, b), converges to $f(x)$ in the ordinary sense if

$$\lim_{n\to\infty} f_n(x) = f(x). \tag{1}$$

A sequence of functions $f_n(x)$ converges in the mean with index $p>0$ to $f(x)$ if

$$\lim_{n\to\infty} \int_a^b |f_n(x) - f(x)|^p \, dx = 0. \tag{2}$$

Throughout the discussion we assume the integration to be taken in the Riemann sense.

Convergence in the mean does not imply ordinary convergence. A familiar example* is the following. Consider the closed intervals $(0, 1/2)$, $(1/2, 1)$, $(0, 1/3)$, $(1/3, 2/3)$, $(2/3, 1)$, $(0, 1/4)$, $\cdots$. Let $f_n(x)=1$ in the nth interval and zero elsewhere. Here $\lim_{n\to\infty} f_n(x)$ does not exist. But the limit in the mean does exist since we have $\lim_{n\to\infty} \int_0^1 |f_n(x)-f(x)|^p\, dx = \lim_{n\to\infty} L_n = 0$ where L_n is the length of the nth interval and $f(x)=0$.

However ordinary convergence, uniform or non-uniform, does imply mean convergence under the conditions as stated in the following theorem.

THEOREM. *If in the finite closed interval (a, b) $f_n(x)$ is bounded in both n and x, if $f_n(x)$ and $f(x)$ are Riemann integrable, if $|f_{n+1}-f(x)| \leqq |f_n(x)-f(x)|$, and if $\lim_{n\to\infty} f_n(x)=f(x)$, then $\lim_{n\to\infty} \int_a^b |f_n(x)-f(x)|^p\, dx=0$ $(p>0)$.*

Proof. Let $F_n(x)=f_n(x)-f(x)$, and let $|f_n(x)| \leqq M/2$ so that $|f(x)| \leqq M/2$. Then we have $|F_n(x)| \leqq M$ for all x and n, and the $\lim_{n\to\infty} F_n(x)=0$. Let $I_n = \int_a^b |F_n(x)|^p\, dx$. We must show $\lim_{n\to\infty} I_n=0$. For a fixed $\epsilon>0$ and an n we have either $|F_n(x)| < \epsilon$ or $\epsilon \leqq |F_n(x)| \leqq M$. These two inequalities divide (a, b) into two sets of intervals. We may ignore the isolated singularities of $F_n(x)$ in making this subdivision since they do not affect the value of the integral. Thus we

* Titchmarsh: The Theory of Functions, Art. 12.53

have $I_n<(b-a)\epsilon^p+M^pB_n$ where B_n is the sum of the lengths of all intervals in which $|F_n(x)|\geqq\epsilon$. But the $\lim_{n\to\infty} B_n=0$ as may be seen by considering the sequence, $B_1, B_2, B_3, \cdots, B_n, \cdots$. We necessarily have $B_1\geqq B_2\geqq B_3\geqq \cdots \geqq B_n \geqq \cdots$ in view of the fact that $|F_{n+1}(x)|\leqq|F_n(x)|$. Such a positive sequence must necessarily approach a unique limit since it is monotonic, is bounded above by $(b-a)$, and is bounded below by zero. Let this limit be equal to $B\neq 0$. Then we have $|F_n(x)|\geqq\epsilon$ for all n in intervals whose sum is B. This means there are points where $\lim_{n\to\infty}|F_n(x)|\neq 0$, which is contrary to hypothesis. Thus $B=0$. Hence I_n may be made as small as desired, and the theorem is proved. The theorem may be extended to sequences which are not monotonic provided the intervals over which the sequences are not monotonic can be made as small as desired. This is actually the case in the first example below.

Example 1. The function $f_n(x)=n^c x^{1/2}\exp(-n^2x^2/4)$ in $(0, 1)$ with $f(x)=0$ illustrates the behavior of the two types of convergence. The $\lim_{n\to\infty} f_n(x)$ is zero everywhere for all c. The maximum value of $f_n(x)$ occurs at $x=1/n$ where we have $f_n(1/n)=n^{c-1/2}\exp(-1/4)$. The convergence of the sequence $f_n(x)$ is uniform for $c<1/2$. Our theorem now tells us we have mean convergence when $c\leqq 1/2$ providing we note the comment at the end of the proof above; for we have $f_n(x)\geqq f_{n+1}(x)\geqq f_{n+2}(x)\geqq \cdots$ except in an interval of length less than $1/n$. Now by direct integration we may verify mean square convergence ($p=2$), for we have $\int_0^1|f_n(x)-f(x)|^2\,dx=n^{2c-2}(1-\exp(-n^2/2))$. The table below shows the various possibilities for $\max f_n(x)$ and mean square convergence as c increases.

	$\lim_{n\to\infty} f_n(1/n)$	$\lim_{n\to\infty}\int_0^1 \lvert f_n(x)-f(x)\rvert^2dx$
$c<1/2$	0	0
$c=1/2$	$\exp(-1/4)$	0
$1/2<c<1$	∞	0
$c=1$	∞	1
$c>1$	∞	∞

Example 2. Let $f_n(x)=\exp(nx^2)/(1+\exp(nx^2))$ in $(-1, 1)$ where $f(0)=1/2$ and $f(x)=1$ when $x\neq 0$. Both $f_n(x)$ and $f(x)$ satisfy the hypothesis of the theorem above, and hence we have mean convergence. Here the "lim" cannot pass under the integral sign because the integrand does not converge uniformly; also, it is impossible to perform the integration by elementary methods.

We may still have mean convergence when $f_n(x)$ is unbounded or when (a, b) is an infinite interval. If we consider mean square convergence, always we must have the area bounded by $[F_n(x)]^2$ and the x-axis arbitrarily small. For example, the function $f_n(x)$ may be unbounded in n as in Ex. 1 above when $1>c>1/2$, unbounded in x (as in the sequence $(nx)^{-1/3}$ in $(-1, 1)$ where $f(0)=\infty$ and $f(x)=0$ for $x\neq 0$), or unbounded in both x and n (as in a suitable combination of the two preceding cases). On the other hand consider $f_n(x)=(nx)^{-3/5}$. This sequence has the same limit function as $(nx)^{-1/3}$ above; the integral $\int_{-1}^{1} f_n(x)\,dx$ converges absolutely as an improper integral; but $f_n(x)$ does not converge to $f(x)$ in the mean square sense in $(-1, 1)$.

PROOFS OF THE ADDITION FORMULAE FOR SINES AND COSINES

A condensation by the editor of independent papers by

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The standard proof of the addition formulae for sines and cosines is certainly one of the least satisfactory sections of the usual trigonometry text. Its chief failings are its complexity, the artificiality of the construction required, and the limitation of the magnitudes of the angles involved. Two alternative proofs of these formulae are published below. The first, submitted by L. J. Burton, assumes a very elementary knowledge of analytic geometry. This proof has been taught in several universities for some time and probably has a long history; but

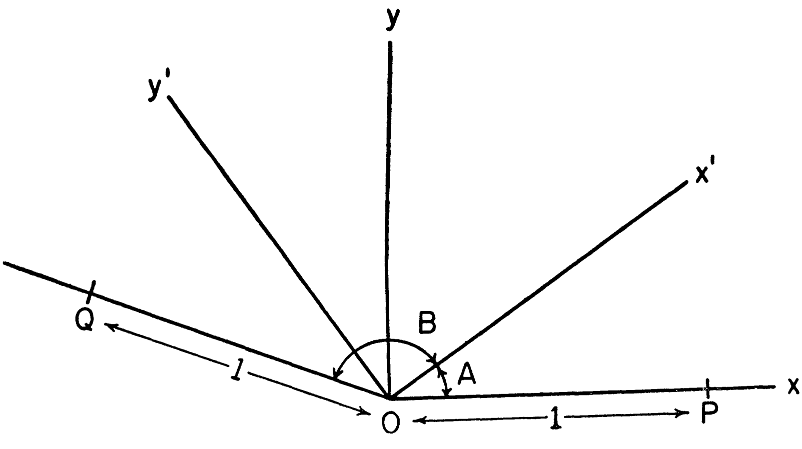

FIG. 1

since it does not appear in the popular textbooks and is unknown to most teachers, it seems desirable to make it more generally available. It has the advantage of placing no restriction on the size of the angles in addition to greater simplicity. The second, submitted by E. A. Hedberg, is based upon the laws of sines and cosines and assumes no analytic geometry. Since it makes use of triangles, it is valid only in case all of the angles involved are less than 180 degrees.