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Unification and infinite series

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Unification and infinite series

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Some infinite series are analysed on the basis of the hypergeometric function and integer structure and modular rings. The resulting generalized functions are compared with differentiation of the 'mother' series.

Keywords: unification; infinite series; hypergeometric

1. Introduction

Infinite series have intrigued mathematicians for centuries, as has the concept of unification, whereby seemingly unrelated functions can be shown to be particular cases of some general function. A case in point occurred early in the nineteenth century when many known functions were shown to be particular cases of the hypergeometric function [1]. Another unifying approach is to analyse systems using integer structure (IS) and modular rings [2]. Here, we illustrate how some series may be thus interpreted, both for their mathematical interest and pedagogical value, so far as they link a variety of ideas.

2. Hypergeometric functions

Many functions can be represented by infinite series. For example, for $-1 < x \leq 1$,

$$\begin{aligned}\ln(1+x) &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots \\ &= xF(1, 1; 2; -x),\end{aligned} \tag{2.1}$$

is a hypergeometric function, defined in general by

$$\begin{aligned}F(a, b; c; x) &= 1 + \frac{ab}{c}x + \frac{a(a+1)b(b+1)}{2c(c+1)}x^2 + \frac{a(a+1)(a+2)b(b+1)(b+2)}{3!c(c+1)(c+2)}x^3 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{a^{\bar{n}} b^{\bar{n}}}{c^{\bar{n}}} \frac{x^n}{n!},\end{aligned} \tag{2.2}$$

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in which a, b, c are rationals and x is a variable, and where $a^{\overline{n}}$ is the rising factorial coefficient [3]

$$a^{\overline{n}} = a(a+1)(a+2)\cdots(a+n-1), \quad n > 0, \; a^{\overline{0}} = 1, \tag{2.3}$$

in contrast to $a^{\underline{n}}$, the more commonly used falling factorial coefficient defined by

$$a^{\underline{n}} = a(a-1)(a-2)\cdots(a-n+1), \quad n > 0, \; a^{\underline{0}} = 1.$$

Carlitz [4] studied hypergeometric differential equations and it suggests the following explorations. From (2.1),

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\ln(1+x) &= (1+x)^{-1}\\ &= 1 - x + x^2 - x^3 + \cdots\\ &= F(1,1;1;-x). \qquad (2.4)\end{aligned}$$

And, if we restrict x by $|x| < 1$, then

$$\frac{\mathrm{d}}{\mathrm{d}x}[xF(1,1;2;-x)] = F(1,1;2;-x) + xF'(1,1;2;-x) \tag{2.5}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)}x^n + \sum_{n=1}^{\infty}\frac{(-1)^n n}{(n+1)}x^n \tag{2.6}$$

$$= \sum_{n=0}^{\infty}(-1)^n x^n \tag{2.7}$$

$$= (1+x)^{-1} \tag{2.8}$$

$$= F(1,1;1;-x). \tag{2.9}$$

Next, from (2.4)

$$-\frac{\mathrm{d}}{\mathrm{d}x}F(1,1;1;-x) = -\frac{\mathrm{d}}{\mathrm{d}x}(1+x)^{-1}$$

$$= (1+x)^{-2} \tag{2.10}$$

$$= 1 - 2x + 3x^2 - 4x^3 + \cdots$$

$$= \begin{cases} F(2,1;1;-x)\\ F(1,2;1;-x)\end{cases} \tag{2.11}$$

Furthermore, we have from (2.5) that

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}[xF(1,1;2;-x)] = 2F'(1,1;2;-x) + xF''(1,1;2;-x) \tag{2.12}$$

$$= \left(1 - \frac{4}{3}x + \frac{6}{4}x^2 - \cdots\right) + \left(-\frac{2}{3}x + \frac{3}{2}x^2 - \frac{8}{5}x^3 + \cdots\right) \tag{2.13}$$

$$= 1 - 2x + 3x^2 - 4x^3 + \cdots \tag{2.14}$$

$$= (1+x)^{-2} \tag{2.15}$$

$$= \begin{cases} F(2,1;1;-x)\\ F(1,2;1;-x).\end{cases} \tag{2.16}$$

Or the hypergeometric differential equation

$$0 = \frac{d^2}{dx^2}F(1.1 : 2; -x) + \frac{d}{dx}F(1, 1; 1; -x).$$

Similarly, students might like to study

$$\frac{d}{dx}(1 + x)^{-2} = -2(1 + x)^{-3}, \tag{2.17}$$

Another example worth exploring by students is

$$\begin{aligned} \tan^{-1} x &= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(1/2)^{\overline{n}} 1^{\overline{n}}}{(3/2)^{\overline{n}}} \frac{x^{2n}}{n!} \\ &= xF\left(\frac{1}{2}, 1; \frac{3}{2}; -x^2\right). \end{aligned} \tag{2.18}$$

3. IS analysis

Another way to interpret infinite series is by means of IS analysis [2]. Here we use the modular ring Z_4, where the integers, I, fall into four classes such that

$$I = 4r_i + i \tag{3.1}$$

where r refers to the rows and i to the classes (remainders modulo 4) in the schematic representation in Table 1. Students can check that $\overline{2}_4$ has no odd powers and $\overline{3}_4$ has no even powers.

Consider ln(1+x) with $x = 1$:

$$(\ln 2) - 1 = -\frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots \tag{3.2}$$

Table 1. Rows with the modular ring Z_4.

$r\downarrow$ $i\rightarrow$	$\overline{0}_4$	$\overline{1}_4$	$\overline{2}_4$	$\overline{3}_4$
0	0	1	2	3
1	4	5	6	7
2	8	9	10	11
3	12	13	14	15
4	16	17	18	19
5	20	21	22	23
6	24	25	26	27
7	28	29	30	31
…		…		

The denominators of these fractions belong to the sequence

$$\{\overline{2}_4\,\overline{3}_4\,\overline{0}_4\,\overline{1}_4\} \equiv \{4r_2+2, 4r_3+3, 4r_0, 4r_1+1\}.$$

That is, since $r_0 = r_1 = r$ and $r_2 = r_3 = r-1$,

$$\begin{aligned}(\ln 2) - 1 &= \sum_{r=1}^{\infty} -\frac{16r^2 - 4r + 1}{4r(16r^2-1)(2r-1)} \\ &= -\frac{13}{60} - \frac{57}{1512} - \frac{133}{8580} - \cdots \\ &= -\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5}\right) - \left(\frac{1}{6} - \frac{1}{7} + \frac{1}{8} - \frac{1}{9}\right) - \cdots \end{aligned} \tag{3.3}$$

Again, from (2.18) we see that the denominators (and indices) of $\tan^{-1}x$ belong to the rather neat sequence

$$\left\{\overline{1}_4\,\overline{3}_4\,\overline{1}_4\,\overline{3}_4\right\} \equiv \left\{(4r+1)(4r+3)(4(r+1)+1)(4(r+1)+3)\cdots\right\} \tag{3.4}$$

so that we can form the sum

$$\begin{aligned}\tan^{-1} x &= \sum_{r=0}^{\infty} \frac{(4r(1-x^2) + (3-x^2))x^{4r+1}}{16r(r+1)+3} \\ &= \left(x - \frac{1}{3}x^3\right) + \left(\frac{1}{5}x^5 - \frac{1}{7}x^7\right) + \cdots .\end{aligned} \tag{3.5}$$

4. Concluding comments

The special functions in general, but hypergeometric functions in particular, lend themselves to undergraduate project work, outside the main standard syllabus but close enough to it for genuine enrichment for both the applied mathematics student [5] and the pure mathematics student [6].

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Technology and x^x

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The revolution of technology in education has heralded a new wave of learning styles and in some cases total dependence on technology. Yet, statistics reveal no significant improvement in student performance in Mathematics and a downward trend in basic algebra skills. The major impediment in the learning process is lack of understanding of the concepts and a virtual total dependence on technology. We expose the dangers of total dependence on technology by using the much-ignored function $f(x) = x^x$ as a case study and conclude that technology can be a valuable tool provided the student has complete understanding of related concepts.

Keywords: graphing calculator; domain; limit

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1. Introduction

The12th-grade results from the Third International Mathematics and Science Series show that US students not only scored below the international average, it reported that these students scored below all but 2 of the 20 nations reported [1]. This low ranking occurred despite the widespread availability of technology both in the classroom and at home. According to the American Management Association annual survey on workplace testing released in 2001 [2], the leading membership-based management development organization in the world, more than a third of job applicants tested in reading and mathematics in 2000 lacked the basic skills necessary to perform the jobs they sought. This article will examine some of the positive ways a graphics calculator can be used to foster student understanding, learning and exploration while also pointing out the limitations of this tool and the resulting need to understand basic definitions and think analytically.

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