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On a general class of trigonometric functions and Fourier series†

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We discuss a general class of trigonometric functions whose corresponding Fourier series can be used to calculate several interesting numerical series. Particular cases are presented.

Keywords: Fourier series; sums; numerical series

1. Introduction

In a recent paper [1], where we presented a note regarding a sum involving odd numbers, we discussed a calculation by considering a general alternating numerical series; as a by-product we recovered a result presented in the Trenčevski's paper [2] and pointed out a possible generalization of the results.

In a more recent paper [3], we presented a general formula for a triple product involving four real numbers, and as a particular case, we got the sum of a triple product with four odd numbers. A general formula for more than four odd numbers was also derived. All those results are calculated using only concepts of standard analysis, particularly numerical series and partial fractions. Some of the results were presented in terms of the beta function [4].

Numerical series can also be obtained as special cases of convenient Fourier series expansions, i.e. one may calculate the sum associated with a numerical series by means of the corresponding Fourier series expansions. Using this argument we have shown the advantage of a Fourier expansion over some other representations, particularly Frobenius series, for example, and discuss how to calculate a Fourier series associated with a class of trigonometric functions [5]. Thus, using Fourier series and the corresponding Parseval identity we recover, in a different way, the same results presented in references [1,3]. Moreover, we derive several general results involving numerical series which cannot be found, for instance, in reference [4].

In this article, we discuss the most general case involving a calculation of the Fourier series associated with a convenient class of trigonometric functions. With our formulas one can derive particular products involving odd numbers; recover the results presented in [5], particularly the triple and fourth product involving odd integers; and derive several interesting results associated with particular numerical series which cannot be found in references [4,6].

The article is organized as follows: In Section 2, we present a convenient class of trigonometric functions, we demonstrate two theorems associated with the corresponding Fourier series and obtain our main result; in Section 3, we discuss particular cases and we show some results involving sums associated with the numerical series, and

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†Dedicated to Prof. J. Bellandi Filho, our teacher and advisor, on his 65th birthday.

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the Parseval identity is used to obtain an interesting result. Finally, we present our concluding remarks.

2. A class of trigonometric functions

In this section, we present a class of trigonometric functions which we shall discuss in association with a convenient Fourier series expansion [7].

There are several trigonometric series that are not Fourier series. As an example, we cite the function $f(x) = \cos^2 x$, with $-\pi/2 \le x \le \pi/2$, an even and periodic function whose expansion in a Fourier series allows us to write $f(x) = (1 + \cos 2x)/2$, i.e. in this case, a well-known trigonometric identity involving the double arc. On the other hand, the same function can be expressed on the same interval as a convergent trigonometric series that is not a Fourier series [5].

We consider a class of trigonometric functions, the integer power of the cosine function.[1] The expansion of a power of cosine in terms of a multiple arc can be separated in even power and odd power, respectively, as follows [6]:

$$\cos^{2n} x = \frac{1}{2^{2n}} \left\{ \sum_{k=0}^{n-1} 2 \binom{2n}{k} \cos[2(n-k)x] + \binom{2n}{n} \right\} \tag{1}$$

and

$$\cos^{2n+1} x = \frac{1}{2^{2n}} \left\{ \sum_{k=0}^{n} \binom{2n+1}{k} \cos\left[2\left(n - k + \frac{1}{2} \right) x \right] \right\} \tag{2}$$

with $n = 1, 2, 3, \ldots, N$.

Theorem 1: *The periodic trigonometric function $f(x) = \cos^{2n} x$, $n = 0, 1, 2, 3, \ldots, N$, with period $p = \pi$, expanded in a Fourier series, on the interval $-\pi/2 \le x \le \pi/2$ reveals an identity only.*

Proof: To prove this theorem, we first have to calculate the corresponding Fourier coefficients. We note that the function is even and thus the Fourier coefficients b_m are equal to zero for all $m = 1, 2, 3, \ldots$ Then we have to calculate only the coefficients

$$a_0 = \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} f(x) \mathrm{d}x \quad \text{and} \quad a_m = \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} f(x) \cos(2mx) \mathrm{d}x, \tag{3}$$

with $m = 1, 2, 3, \ldots$

Introducing Equation (1) in Equation (3) we obtain, respectively,

$$a_0 = \frac{2}{2^{2n}} \binom{2n}{n} \tag{4}$$

and[2]

$$a_m \equiv a_{n-k} = \frac{2}{2^{2n}} \sum_{k=0}^{n-1} \binom{2n}{k}. \tag{5}$$

Substituting Equations (4) and (5) in the Fourier expansion, we get[3]

$$
\begin{aligned}
f(x) \equiv \cos^{2n} x &= \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(2mx) \\
&= \frac{1}{2^{2n}} \binom{2n}{n} + \frac{2}{2^{2n}} \sum_{k=0}^{n-1} \binom{2n}{k} \cos[2(n-k)x] \\
&= \frac{1}{2^{2n}} \left\{ \binom{2n}{n} + \sum_{k=0}^{n-1} 2 \binom{2n}{k} \cos[2(n-k)x] \right\},
\end{aligned} \tag{6}
$$

which is the same as Equation (1), i.e. an identity □

In such cases, in order to obtain a convenient Fourier expansion, i.e. an expression that is not just a trigonometric identity, we must extend the trigonometric function on a particular interval.

Theorem 2: *The periodic trigonometric function $f(x) = \cos^{2n+1} x$, $n = 0, 1, 2, 3, \ldots, N$, with period $p = \pi$, expanded in a Fourier series on the interval $-\pi/2 \le x \le \pi/2$ has Fourier coefficients*

$$
a_0 = \frac{4}{\pi\, 2^{2n}} \sum_{k=0}^{n} \binom{2n+1}{k} \frac{(-1)^{n-k}}{2n-2k+1}
$$

and[4]

$$
a_m = \frac{4}{\pi\, 2^{2n}} \sum_{k=0}^{n} \binom{2n+1}{k} (-1)^{n-k+m} \frac{2k-2n-1}{4m^2 - (2k-2n-1)^2},
$$

with $m = 1, 2, 3, \ldots$

Proof: In this case, the trigonometric function $f(x) = \cos^{2n+1} x$, $n = 0, 1, 2, \ldots, N$ is also an even function, so that $b_m = 0$ for all $m = 1, 2, 3, \ldots$ Here we show only the first result, i.e. the explicit calculation of a_0. To this end we must calculate the integral

$$
a_0 = \frac{4}{\pi} \int_0^{\pi/2} \frac{1}{2^{2n}} \sum_{k=0}^{n} \binom{2n+1}{k} \cos[(2n-2k+1)x] \mathrm{d}x
$$

and after a simple integration, we get

$$
a_0 = \frac{4}{\pi\, 2^{2n}} \sum_{k=0}^{n} \binom{2n+1}{k} \frac{(-1)^{n-k}}{2n-2k+1}
$$

which is the Fourier coefficient for $m = 0$.

To obtain a_m, $m = 1, 2, 3, \ldots$, we must calculate a similar integral, i.e.

$$
a_m = \frac{4}{\pi} \int_0^{\pi/2} \frac{1}{2^{2n}} \sum_{k=0}^{n} \binom{2n+1}{k} \cos[(2n-2k+1)x] \cos(2mx) \mathrm{d}x.
$$

Using a trigonometric identity involving a product of two cosines and simple integration, we obtain the equations for the Fourier coefficients a_m, with $m = 1, 2, 3, \ldots$ See Equation (7) below. □

We conclude this section pointing out that the Fourier series associated with the trigonometric function $f(x) = \cos^{2n} x$ on the interval $-\pi/2 \le x \le \pi/2$, periodic with period π, produces only a well-known trigonometric identity, while for the same interval the Fourier expansion for the trigonometric function $f(x) = \cos^{2n+1} x$ yields

$$\cos^{2n+1} x = \frac{2}{\pi\, 2^{2n}} \sum_{k=0}^{n} \binom{2n+1}{k} \frac{(-1)^{n-k}}{2n-2k+1} + \sum_{m=1}^{\infty} \left\{ \frac{4}{\pi\, 2^{2n}} \sum_{k=0}^{n} \binom{2n+1}{k} \frac{(-1)^{n-k+m}(2k-2n-1)}{4m^2-(2k-2n-1)^2} \right\} \cos(2mx) \tag{7}$$

with $n = 0, 1, 2, \ldots, N$ and $-\pi/2 \le x \le \pi/2$.

We also mention that, if we change the order of the sums in the right-hand side of Equation (7) and perform the infinite sum, using the result [4]

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2-a^2} \cos kx = \frac{\pi}{2a} \cos ax \csc \pi a - \frac{1}{2a^2},$$

with $-\pi \le x \le \pi$, we can recover Equation (2). Nevertheless, we do not take this way, as will seen below.

3. Particular cases

As mentioned in the last section, we will not change the order of sums in order to perform the infinite one. For our purposes, it is better to work with Equation (7).

First, taking $n = 1$ in Equation (7), we recover all results obtained in [5]. Also, using Equation (7), for a fixed value of n, we can produce a large class of numerical series whose sums are known. Here we present some new sums obtained by considering particular values of the independent variable, for the case $n = 2$ only.

Thus, putting $n = 2$ into Equation (7) we get an expansion of the trigonometric function $f(x) = \cos^5 x$, with $-\pi/2 \le x \le \pi/2$ in a Fourier series, periodic with period π, as follows:

$$\cos^5 x = \frac{16}{15\pi} - \frac{480}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^m}{(4m^2-25)(4m^2-9)(4m^2-1)} \cos(2mx). \tag{8}$$

Using this expression with $x = 0$, $x = \pi/4$ and $x = \pi/2$, we obtain sums for the corresponding numerical series.

3.1. *The case $x = 0$*

Putting $x = 0$ into Equation (8) and after a few simple manipulations, we get

$$\sum_{m=1}^{\infty} \frac{(-1)^m}{(4m^2-25)(4m^2-9)(4m^2-1)} = \frac{16-15\pi}{7200}.$$

After a simple rearrangement we obtain an interesting alternating numerical series involving a product of six odd numbers, as follows:

$$\frac{1}{1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11}-\frac{1}{3\cdot 5\cdot 7\cdot 9\cdot 11\cdot 13}+\frac{1}{5\cdot 7\cdot 9\cdot 11\cdot 13\cdot 15}-\cdots=\frac{315\pi-976}{151200}.$$

3.2. *The case $x = \pi/4$*

In this case putting $x=\pi/4$ into Equation (8), we obtain

$$\sum_{m=1}^{\infty}\frac{(-1)^m}{(16m^2-25)(16m^2-9)(16m^2-1)}=\frac{128-15\pi\sqrt{2}}{57600}.$$

3.3. *The case $x = \pi/2$*

Putting $x=\pi/2$ into Equation (8), we have

$$\sum_{m=1}^{\infty}\frac{1}{(4m^2-25)(4m^2-9)(4m^2-1)}=\frac{1}{450}.$$

4. Parseval identity

As we already know, the Parseval identity, which relates the Fourier coefficients, is useful also to calculate several definite integrals or sums [5]. Here we use the Parseval identity, always associated with the case $n=2$, to calculate the sum of another numerical series.

Using Equation (8) and the Parseval identity, we obtain the following expression

$$\sum_{m=1}^{\infty}\frac{1}{(4m^2-25)^2(4m^2-9)^2(4m^2-1)^2}=\frac{14175\pi^2-65536}{6635520000}.$$

In order to get this expression, we have also used the definition of gamma function [6].

Finally, as a by-product we can rearrange this expression to obtain the following interesting infinity sum

$$\frac{1}{(1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11)^2}+\frac{1}{(3\cdot 5\cdot 7\cdot 9\cdot 11\cdot 13)^2}+\frac{1}{(5\cdot 7\cdot 9\cdot 11\cdot 13\cdot 15)^2}+\cdots$$
$$=\frac{6251175\pi^2-61669376}{2926264320000}.$$

5. Concluding remarks

We have considered a convenient class of trigonometric functions $f(x)=\cos^k x$, for k a positive integer, which are even functions for any value of k. One can put itself the question, why not the function $g(x)=\sin^k x$ in lieu of $f(x)$? The simple anwser is: if k is an even number, $g(x)$ is always an even function but in the case where k is an odd number, we have an odd function, conversely, to the function $f(x)$, which is always an even function. For this end, the function $g(x)$ as given above cannot represent a convenient class of trigonometric function, in this case, i.e. on the considered interval.

We discussed the calculation of the Fourier series associated to a class of even trigonometric functions $f(x) = \cos^k x$, where k is a positive integer. With the resulting expressions we derived, for the case $n = 2$, particular series of products of odd numbers, recovered some recent results involving triple and quadruple products of odd integers and evaluated several interesting sums associated with particular numerical series.

Finally, we conjecture that the Fourier expansion for our convenient function, $f(x)$, can be related to the Riemann zeta function, $\zeta(z)$, in the sense that we have $\zeta(-2m) = 0$ and $\zeta(1 - 2m) \neq 0$ for $m = 1, 2, 3, \ldots$ [8,9]. We will discuss this problem in a forthcoming paper [10].

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Notes

1. We also note that another possible class of functions is $g(x) = \sin^\ell x$ with $\ell = 1, 2, 3, \ldots$ expanded on the interval $a \leq x \leq b$ where a and b must be chosen in a convenient way. Here, we do not discuss this case.
2. In the calculation of the Fourier coefficient a_m, we have two possibilities. One of them produces the value $m = k - n$; as $n > k$ and m is a positive integer, this case must be omitted. The other one produces $m = n - k$ and then we have $a_m \equiv a_{n-k}$.
3. Note that we do not have an infinite series. The unique term contributing to the sum is $m = n - k$.
4. Note that the equation for a_m gives the same a_0 for $m = 0$.

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Using the Hill cipher to teach cryptographic principles

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The Hill cipher is the simplest example of a *block cipher*, which takes a block of plaintext as input, and returns a block of ciphertext as output. Although it is insecure by modern standards, its simplicity means that it is well suited for the teaching of such concepts as encryption modes, and properties of cryptographic hash functions. Although these topics are central to modern cryptography, it is hard to find good simple examples of their use. The conceptual and computational simplicity of the Hill cipher means that students can experiment with these topics, see them in action, and obtain a better understanding that would be possible from a theoretical discussion alone. In this article, we define the Hill cipher and demonstrate its use with different modes of encryption, and also show how cryptographic hash functions can be both designed and broken. Finally, we look at some pedagogical considerations.

Keywords: cryptography; teaching; Hill cipher

1. Basic cryptography

For this article, cryptography will be taken to mean the algorithms, software and hardware necessary to transform a message or data, with a key, into data which is unreadable or unrecoverable without using the key. We assume the following definitions:

Plaintext: This is the message or original data.
Key: A piece of information or data which are used as part of the encryption process.
Encryption: This is transformation of the message, with the key, so that the result is unreadable without the key.
Ciphertext: This is the result when the encryption routine is applied to the plaintext.
Decryption: This is the process of undoing the encryption to turn the ciphertext back into the original plaintext.

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