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# Alternative proofs for inequalities of some trigonometric functions 

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By using an identity relating to Bernoulli's numbers and power series expansions of cotangent function and logarithms of functions involving sine function, cosine function and tangent function, four inequalities involving cotangent function, sine function, secant function and tangent function are established.

Keywords: inequality; power series expansion; tangent function; secant function; cosecant function; sine function; Bernoulli's number

## 1. Introduction

The Bernoulli's numbers $B_{n}$ and Euler's numbers $E_{n}$ for nonnegative integers $n$ are, repectively, defined in [1,7] and [20, p. 1 and p. 6] by

$$
\begin{equation*}
\frac{t}{e^{t}-1}+\frac{t}{2}=1+\sum_{n=0}^{\infty}(-1)^{n-1} B_{n} \frac{t^{2} n}{(2 n)!}, \quad|t|<2 \pi \tag{1}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{2 e^{t / 2}}{e^{t}+1}=\sum_{n=0}^{\infty} \frac{(-1)^{n} E_{n}}{(2 n)!}\left(\frac{t}{2}\right)^{2 n}, \quad|t|<\pi \tag{2}
\end{equation*}
$$

The following power series expansions are well known and can be found in [1] and [7, pp. 227-229]:

$$
\begin{gather*}
\cot x=\frac{1}{x}-\sum_{k=1}^{\infty} \frac{2^{2 k} B_{k}}{(2 k)!} x^{2 k-1}, \quad 0<|x|<\pi,  \tag{3}\\
\ln \frac{\sin }{x}=-\sum_{k=1}^{\infty} \frac{2^{2 k-1} B_{k}}{k(2 k)!} x^{2 k}, \quad 0<|x|<\pi,  \tag{4}\\
\ln \cos x=-\sum_{k=1}^{\infty} \frac{2^{2 k-1}\left(2^{2 k}-1\right) B_{k}}{k(2 k)!} x^{2 k}, \quad|x|<\frac{\pi}{2},  \tag{5}\\
\ln \frac{\tan x}{x}=\sum_{k=1}^{\infty} \frac{2^{2 k}\left(2^{2 k-1}-1\right) B_{k}}{k(2 k)!} x^{2 k}, \quad 0<|x|<\frac{\pi}{2} . \tag{6}
\end{gather*}
$$

It is also well known [7, p. 231] that

$$
\begin{equation*}
\sum_{m=1}^{\infty} \frac{1}{m^{2 n}}=\frac{\pi^{2 n} 2^{2 n-1}}{(2 n)!} B_{n} \tag{7}
\end{equation*}
$$

[^0]The Becker-Stark's inequality ([2], [13, p. 156] and [11]) states that for $0<x<1$,

$$
\begin{equation*}
\frac{4}{\pi} \cdot \frac{x}{1-x^{2}}<\tan \frac{\pi x}{2}<\frac{\pi}{2} \cdot \frac{x}{1-x^{2}} \tag{8}
\end{equation*}
$$

For $x \in(0, \pi / 6)$, Djokvie's inequality states [11] that

$$
\begin{equation*}
x+\frac{1}{3} x^{3}<\tan x<x+\frac{4}{9} x^{3} . \tag{9}
\end{equation*}
$$

In [3], the following inequalities are proved: For $x \in(0, \pi / 2)$ and $n \in 1 N$,

$$
\begin{equation*}
\frac{2^{2(n+1)}\left(2^{2(n+1)}-1\right) B_{n+1}}{(2 n+2)!} x^{2 n} \tan x<\tan x-S_{n}(x)<\left(\frac{2}{\pi}\right)^{2 n} x^{2 n} \tan x \tag{10}
\end{equation*}
$$

where

$$
\begin{equation*}
S_{n}(x)=\sum_{i=1}^{n} \frac{2^{2 i\left(2^{2 i}-1\right) B_{i}}}{(2 i)!} x^{2 i-1} \tag{11}
\end{equation*}
$$

If taking $n=1$ in (10), for $0<x<(3 / \pi) \sqrt{5\left(\pi^{2}-8\right) / 38}$, the left hand side inequality in (10) is better than the left hand side inequality in (8). If taking $n=2$ in (10), we obtain

$$
\begin{equation*}
x+\frac{1}{3} x^{3}+\frac{2}{15} x^{4} \tan x<\tan x<x+\frac{1}{3} x^{3}+\left(\frac{2}{\pi}\right)^{4} x^{4} \tan x, \quad x \in\left(0, \frac{\pi}{2}\right) \tag{12}
\end{equation*}
$$

The constants $2 / 15$ and $(2 / \pi)^{4}$ in (12) are the best possible. Since

$$
\frac{1}{3}+\left(\frac{2}{\pi}\right)^{4} x \tan x<\frac{1}{3}+\left(\frac{2}{\pi}\right)^{4} \cdot \frac{\pi}{6} \cdot \frac{1}{\sqrt{3}}<\frac{4}{9}
$$

the inequalities in (12) are better than those in (9).
Recently a number of articles have been published on inequalities involving trigonometric functions $[5,6,8-10,15,16,18]$, estimates of remainders of elementary functions $[12,14]$ and related questions [17,19].

The purpose of this article is to give the second proofs of the following four inequalities involving some trigonometric functions, which were established in [4].

Theorem 1: For $0<x<1$,

$$
\begin{equation*}
\frac{2}{\pi} \cdot \frac{x}{1-x^{2}}<\frac{1}{\pi x}-\cot (\pi x)<\frac{\pi}{3} \cdot \frac{x}{1-x^{2}} \tag{13}
\end{equation*}
$$

The constants $2 / \pi$ and $\pi / 3$ in (13) are the best possible.
For $0<|x|<1$, we have

$$
\begin{align*}
& \ln \left(\frac{\pi x}{\sin (\pi x)}\right)<\frac{\pi^{2}}{6} \cdot \frac{x^{2}}{1-x^{2}}  \tag{14}\\
& \ln \left(\sec \frac{\pi x}{2}\right)<\frac{\pi^{2}}{8} \cdot \frac{x^{2}}{1-x^{2}}  \tag{15}\\
& \ln \left(\frac{\tan \pi x / 2}{\pi x / 2}\right)<\frac{\pi^{2}}{12} \cdot \frac{x^{2}}{1-x^{2}} \tag{16}
\end{align*}
$$

The constants $\pi^{2} / 6, \pi^{2} / 8$ and $\pi^{2} / 12$ are the best possible.

Remark 1: Notice that there are a large number of particular inequalities relating to trigonometric functions in [11,13].

## 2. Proof of Theorem 1

The proof of inequality (13): Define for $0<x<1$

$$
\begin{equation*}
f(x)=\frac{1-x^{2}}{x}\left(\frac{1}{\pi x}-\cot (\pi x)\right) \tag{17}
\end{equation*}
$$

Replacing $x$ by $\pi x$ in (3) yields

$$
\begin{equation*}
\cot (\pi x)=\frac{1}{\pi x}-\sum_{k=1}^{\infty} \frac{2^{2 k} \pi^{2 k+1} B_{k}}{(2 k)!} x^{2 k+1}, \quad 0<|x|<1 \tag{18}
\end{equation*}
$$

Substituting (18) into (17) produces

$$
\begin{equation*}
f(x)=\frac{\pi}{3}+\sum_{k=1}^{\infty}\left(\frac{2^{2 k+2} \pi^{2 k+1} B_{k+1}}{(2 k+2)!}-\frac{2^{2 k+2} \pi^{2 k+1} B_{k}}{(2 k)!}\right) x^{2 k} . \tag{19}
\end{equation*}
$$

Using (7), (19) can be rewritten as

$$
f(x)=\frac{\pi}{3}-\frac{2}{\pi} \sum_{k=1}^{\infty} \sum_{n=1}^{\infty}\left(\frac{1}{n^{2 k}}-\frac{1}{n^{2 k+2}}\right) x^{2 k} .
$$

It is easy to see that $\mathrm{f}(\mathrm{x})$ is strictly decreasing, then $2 / \pi=\lim _{x \rightarrow 1} f(x)<f(x)<\lim _{x \rightarrow 0}$ $f(x)=\pi / 3$. Inequality (13) follows.
The proof of inequality (14): Define for $0<x<1$

$$
\begin{equation*}
g(x)=\frac{1-x^{2}}{x^{2}} \ln \frac{\pi x}{\sin (\pi x)} \tag{20}
\end{equation*}
$$

Replacing $x$ by $\pi x$ in (4) yields

$$
\begin{equation*}
\ln \frac{\pi x}{\sin (\pi x)}=\sum_{k=1}^{\infty} \frac{2^{2 k-1} \pi^{2 k} B_{k}}{k(2 k)!} x^{2 k}, \quad 0<|x|<1 \tag{21}
\end{equation*}
$$

Substituting (21) into (20) leads to

$$
\begin{equation*}
g(x)=\frac{\pi^{2}}{6}-\sum_{k=1}^{\infty}\left(\frac{2^{2 k+1} \pi^{2 k+2} B_{k+1}}{(k+1)(2 k+2)!}-\frac{2^{2 k-1} \pi^{2 k} B_{k}}{k(2 k)!}\right) x^{2 k} . \tag{22}
\end{equation*}
$$

Using (7), (22) can be rearranged to

$$
g(x)=\frac{\pi^{2}}{6}-\sum_{k=1}^{\infty}\left(\frac{1}{k} \sum_{n=1}^{\infty} \frac{1}{n^{2 k}}-\frac{1}{k+1} \sum_{n=1}^{\infty} \frac{1}{n^{2 k+2}}\right) x^{2 k} .
$$

It is easy to see that $g(x)$ is strictly decreasing, thus $g(x)<\lim _{x \rightarrow 0} \mathrm{~g}(x)=\pi^{2} / 6$ which is equivalent to (14).

The proof of inequality (15): Define for $0<x<1$

$$
\begin{equation*}
h(x)=\frac{1-x^{2}}{x^{2}} \ln \left(\sec \frac{\pi x}{2}\right) . \tag{23}
\end{equation*}
$$

Replacing $x$ by $\pi x / 2$ in (5) yields

$$
\begin{equation*}
\ln \left(\sec \frac{\pi x}{2}\right)=\sum_{k=1}^{\infty} \frac{\left(2^{2 k}-1\right) \pi^{2 k} k B_{k}}{2 k(2 k)!} x^{2 k}, \quad 0<|x|<1 . \tag{24}
\end{equation*}
$$

Substituting (24) into (23) leads to

$$
\begin{equation*}
h(x)=\frac{\pi^{2}}{8}-\sum_{k=1}^{\infty}\left(\frac{\left(2^{2 k}-1\right) \pi^{2} B_{k}}{2 k(2 k)!}-\frac{\left(2^{2 k+2}-1\right) \pi^{2 k+2} B_{k+1}}{(2 k+2)(2 k+2)!}\right) x^{2 k} . \tag{25}
\end{equation*}
$$

Using (7), (25) can be rewritten as

$$
\begin{equation*}
h(x)=\frac{\pi^{2}}{8}-\sum_{k=1}^{\infty}\left(\frac{2^{2 k}-1}{k 2^{2 k}} \sum_{n=1}^{\infty} \frac{1}{n^{2 k}}-\frac{2^{2 k+2}-1}{(k+1) 2^{2 k+2}} \sum_{n=1}^{\infty} \frac{1}{n^{2 k+2}}\right) x^{2 k} . \tag{26}
\end{equation*}
$$

It is clear that for $k \in \mathbb{N}$

$$
\begin{equation*}
\sum_{n=1}^{\infty} \frac{1}{n^{2 k}}>\sum_{n=1}^{\infty} \frac{1}{n^{2 k+2}} \tag{27}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{2^{2 k}-1}{k 2^{2 k}}>\frac{2^{2 k+2}-1}{(k+1) 2^{2 k+2}} . \tag{28}
\end{equation*}
$$

From (26), (27) and (28), we readily obtain that $h(x)$ is strictly decreasing. Thus $g(x)<\lim _{x \rightarrow 0} g(x)=\pi^{2} / 8$, which is equivalent to (15).
The proof of inequality (16): Define for $0<x<1$

$$
\begin{equation*}
\varphi(x)=\frac{1-x^{2}}{x^{2}} \ln \left(\frac{\tan \pi x / 2}{\pi x / 2}\right) . \tag{29}
\end{equation*}
$$

Replacing $x$ by $\pi x / 2$ in (6) yields

$$
\begin{equation*}
\ln \left(\frac{\tan (\pi x / 2)}{\pi x / 2}\right)=\sum_{k=1}^{\infty} \frac{\left(2^{2 k-1}-1\right) \pi^{2 k} B_{k}}{k(2 k)!} x^{2 k}, \quad 0<|x|<1 . \tag{30}
\end{equation*}
$$

Substituting (30) into (29) gives

$$
\begin{equation*}
\varphi(x)=\frac{\pi^{2}}{12}-\sum_{k=1}^{\infty}\left(\frac{\left(2^{2 k-1}-1\right) \pi^{2 k} B_{k}}{k(2 k)!}-\frac{\left(2^{2 k+1}-1\right) \pi^{2 k+2} B_{k+1}}{(k+1)(2 k+2)!}\right) x^{2 k} . \tag{31}
\end{equation*}
$$

Using (7), (31) can be rewritten as

$$
\begin{equation*}
\varphi(x)=\frac{\pi^{2}}{12}-\sum_{k=1}^{\infty}\left(\frac{2^{2 k-1}-1}{k 2^{2 k-1}} \sum_{n=1}^{\infty} \frac{1}{n^{2 k}}-\frac{2^{2 k+1}-1}{(k+1) 2^{2 k+1}} \sum_{n=1}^{\infty} \frac{1}{n^{2 k+2}}\right) x^{2 k} . \tag{32}
\end{equation*}
$$

Combining (27) and (28) with (32), we see that $\varphi(x)$ is strictly decreasing. Hence $\varphi(x)<\lim _{x \rightarrow 0} \varphi(x)=\pi^{2} / 12$, which is equivalent to (16).

## References

[1] M. Abramowitz and I.A. Stegun (eds.), Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, National Bureau of Standards, Applied Mathematics Series 55, 4th printing, Washington, 1965, 1972.
[2] M. Becker and E.L. Stark, On a hierachy of quolynomial inequalities for tanx, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 602-633 (1978), pp. 133-138.
[3] Ch.-P. Chen and F. Qi, A double inequality for remainder of power series of tangent function, Tamkang J. Math. 34 (2003), no. 3, 351-355. RGMIA Res. Rep. Coll. 5 (2002), suppl., Art. 2. Available online at http://rgmia.vu.edu.au/v5(E).html
[4] Ch.-P. Chen and F. Qi, Inequalities of some trigonometric functions, Univ. Beog. Publ. Elektrotehn. Fak. Ser. Mat. 15 (2004), 71-78. RGMIA Res. Rep. Coll. 6 (2003), no. 3, Art. 2, 419-429; Available online at http://rgmia.vu.edu.au/v6n3.html
[5] Ch.-P. Chen and F. Qi, On two new proofs of Wilker's inequality, Gāoděng Shùxué Yánjiū (Studies in College Mathematics) 5 (2002), pp. 38-39 (Chinese).
[6] Ch.-P. Chen, J.-W. Zhao, and F. Qi, Three inequalities involving hyperbolically trigonometric functions, Octogon Math. Mag. 12 (2004), no. 2, 592-596. RGMIA Res. Rep. Coll. 6 (2003), 437-443; Available online at http://rgmia.vu.edu.au/v6n3.html
[7] Group of Compilation, Handbook of Mathematics, Peoples' Education Press, Beijing, China, 1979 (Chinese).
[8] B.-N. Guo, W. Li, and F. Qi, in Proofs of Wilker's inequalities involving trigonometric functions, Inequality Theory and Applications, Y. J. Cho, J. K. Kim, and S. S. Dragomir, eds., Nova Science Publishers, Hauppauge, NY, 2003, pp. 109-112.
[9] B.-N. Guo, W. Li and F. Qi, On new proofs of inequalities involving trigonometric functions, RGMIA Res. Rep. Coll. 3 (2000), 167-170; Available online at http://rgmia.vu.edu.au/ v3n1.html
[10] B.-N. Guo, et al., On new proofs of Wilker's inequalities involving trigonometric functions, Math. Inequal. Appl. 6(1) (2003), pp. 19-22.
[11] J.-Ch. Kuang, Applied Inequalities, 3rd ed., Shandong Science and Technology Press, Jinan City, Shandong Province, China, 2004 (Chinese).
[12] M. Merkle, Inequalities for residuals of power series: a review, Univ. Beograd. Publ. Elek- trotehn. Fak. Ser. Mat. 6 (1995), pp. 79-85.
[13] J. Pečarić, Nejednakosti, Element, Zagreb, 1996 (Croatia).
[14] F. Qi, A method of constructing inequalities about $e^{x}$, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. 8 (1997), pp. 16-23.
[15] F. Qi, Refinements and extensions of Jordan's and Kober's inequalities, Gōngkē Shùxué, J. Math. Technol. 12 (1996), pp. 98-102 (Chinese).
[16] F. Qi, L.-H. Cui, and S.-L. Xu, Some inequalities constructed by Tchebysheff's integral inequality, Math. Inequal. Appl. 2 (1999), pp. 517-528.
[17] F. Qi and B.-N. Guo, Estimate for upper bound of an elliptic integral, Math. Practice Theory 26 (1996), pp. 285-288 (Chinese).
[18] F. Qi and Q.-D. Hao, Refinements and sharpenings of Jordan's and Kober's inequality, Math. Inform. Q. 8 (1998), pp. 116-120.
[19] F. Qi and Zh. Huang, Inequalities of the complete elliptic integrals, Tamkang J. Math. 29 (1998), pp. 165-169.
[20] Zh.-X. Wang and D.-R. Guo, Téshū Hánshù Gáilìn (Introduction to Special Function), The Series of Advanced Physics of Peking University, Peking University Press, Beijing, China, 2000 (Chinese).

# The implicit function theorem and non-existence of limit of functions of several variables 

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#### Abstract

We use the Implicit Function Theorem to establish a result of non-existence of limit to a certain class of functions of several variables. We consider functions given by quotients such that both the numerator and denominator functions are null at the limit point. We show that the non-existence of the limit of such function is related with the gradient vectors of the numerator and denominator functions. We prove the limit does not exist if the dimension of the vector subspace spanned by the gradient vectors is $\geq 1$.


Keywords: functions of several variables; implicit function theorem; limits

## 1. Introduction

We use the Implicit Function Theorem to establish a result of non-existence of limit to a certain class of functions of several variables. To show that a function $H$ of several variables has no limit as we approach the origin, for example, we are using this to show that there are two different paths towards the origin along which the function $H$ has different limits. Sometimes, it is a hard task finding these paths. Consider, for instance, the following limit (given as an exercise in 3 )

$$
\lim _{(x, y) \rightarrow(0,0)} H(x, y), \quad \text { where } H(x, y)=\frac{x^{3}}{x^{2}+y} .
$$

Along all the paths given by $y=\alpha x$ or $y=\beta x^{2}$, for $\alpha, \beta \in \mathbb{R}$, with $\beta \neq-1$, it is easy to see that the function $H(x, y)$ always tends to zero.

If we look to the zero level sets of the functions $f(x, y)=x^{3}$ and $g(x, y)=x^{2}+y$, we see they intersect only at the origin, that is $\left\{(x, y) \in \mathbb{R}^{2}, f(x, y)=0\right\} \cap\left\{(x, y) \in \mathbb{R}^{2}\right.$, $g(x, y)=0\}=\{(0,0)\}$. Thus, clearly the previous limit does not exist since $H(x, y)$ is not bounded on any punctured neighbourhood of the origin.

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