LISTA 06 - CÁLCULO 1.

01)

(b) $f(x) = \sqrt{3-2x}$; $x_0 = 1$.

Pela definição:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{3-2.(x+h)} - \sqrt{3-2x}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{3-2x-2h} - \sqrt{3-2x}}{h} \times \frac{\sqrt{3-2x-2h} + \sqrt{3-2x}}{\sqrt{3-2x-2h} + \sqrt{3-2x}}$$

$$= \lim_{h\to 0} \frac{3-2x-2h-(3-2x)}{h.(\sqrt{3-2x-2h} + \sqrt{3-2x})} = \lim_{h\to 0} \frac{2\cancel{h}}{\cancel{h}(\sqrt{3-2x-2h} + \sqrt{3-2x})}$$

$$= \frac{2}{2\sqrt{3-2x}} = \frac{1}{\sqrt{3-2x}} \Longrightarrow \boxed{f'(x) = \frac{1}{\sqrt{3-2x}}}$$

eq. da reta tangente ao gráfico de f em $x_0 = 1$:

neste caso, $y_0 = f(x_0) = \sqrt{3-2.(1)} = 1$, e

$m_t = f'(x_0) = f'(1) = \dfrac{1}{\sqrt{3-2.(1)}} = 1$, e portanto,

$$y - y_0 = m_t(x - x_0), \text{ ou seja,}$$

$$y - 1 = 1.(x-1) \Longrightarrow \boxed{y = x}.$$

(c) $f(x) = \cos\left(2\pi x - \frac{\pi}{3}\right)$; em $x_0 = \frac{1}{6}$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\cos\left(2\pi(x+h) - \frac{\pi}{3}\right) - \cos\left(2\pi x - \frac{\pi}{3}\right)}{h}$$

$$= \lim_{h \to 0} \frac{\cos\left(2\pi x + 2\pi h - \frac{\pi}{3}\right) - \cos\left(2\pi x - \frac{\pi}{3}\right)}{h} \; (*)$$

Vamos deduzir uma fórmula para transformar a diferença trigonométrica do numerador num produto. Da Trigonometria temos:

$$\cos(\alpha + \beta) = \cos\alpha \cdot \cos\beta - \operatorname{sen}\alpha \cdot \operatorname{sen}\beta \qquad (\mathrm{I})$$

$$\cos(\alpha - \beta) = \cos\alpha \cdot \cos\beta + \operatorname{sen}\alpha \cdot \operatorname{sen}\beta \qquad (\mathrm{II})$$

Fazendo (I) − (II), obtemos:

$$\cos(\alpha + \beta) - \cos(\alpha - \beta) = -2 \cdot \operatorname{sen}\alpha \cdot \operatorname{sen}\beta .$$

Por fim, denotando $\begin{cases} \alpha + \beta = p \\ \alpha - \beta = q \end{cases}$ vamos achar $\alpha = \frac{p+q}{2}$ e $\beta = \frac{p-q}{2}$.

Portanto, temos a fórmula:

$$\boxed{\cos p - \cos q = -2 \cdot \operatorname{sen}\frac{p+q}{2} \cdot \operatorname{sen}\frac{p-q}{2}} \qquad (\mathrm{III})$$

Assim, usando (III) no cálculo do limite, obtemos:

$$(*) \lim_{h \to 0} \frac{-2 \cdot \operatorname{sen}\left[\frac{2\pi x + 2\pi h - \frac{\pi}{3} + 2\pi x - \frac{\pi}{3}}{2}\right] \cdot \operatorname{sen}\left[\frac{\cancel{2\pi x} + 2\pi h - \frac{\pi}{3} - \cancel{2\pi x} + \frac{\pi}{3}}{2}\right]}{h}$$

$$= \lim_{h \to 0} \frac{-2 \cdot \operatorname{sen}\left(2\pi x + \pi h - \frac{\pi}{3}\right) \cdot \operatorname{sen}(\pi h)}{h} \times \frac{\pi}{\pi} =$$

$$= \lim_{h \to 0} -2 \operatorname{sen}\left(2\pi x + \pi h - \frac{\pi}{3}\right) \underbrace{\frac{\operatorname{sen}(\pi h)}{\pi h}}_{\xrightarrow[h \to 0]{} 1} \cdot \pi = -2\pi \operatorname{sen}\left(2\pi x - \frac{\pi}{3}\right)$$

$$\Longrightarrow \boxed{f'(x) = -2\pi \operatorname{sen}\left(2\pi x - \frac{\pi}{3}\right)}$$

Vamos agora determinar a eq. da reta tangente ao gráfico de f em $x_0 = \frac{1}{6}$. Neste caso,

$$y_0 = f(x_0) = \cos\left(2\pi \cdot \frac{1}{6} - \frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3} - \frac{\pi}{3}\right) = \cos 0 = 1;$$

e $$m_t = f'(x_0) = f'\left(\frac{1}{6}\right) = -2\pi \cdot \operatorname{sen}\left(2\pi \cdot \frac{1}{6} - \frac{\pi}{3}\right) = 0;$$

e assim:

$$y - y_0 = f'(x_0)(x - x_0)$$

$$\Rightarrow y - 1 = 0 \cdot \left(x - \frac{1}{6}\right) \Rightarrow \boxed{y = 1}$$ eq. da reta tangente ao gráfico de f em $\left(\frac{1}{6}, 1\right)$.

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02) (d) $f(x) = \ln(3-2x)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\ln(3-2(x+h)) - \ln(3-2x)}{h}$$

$$= \lim_{h \to 0} \frac{\ln(3-2x-2h) - \ln(3-2x)}{h} = \lim_{h \to 0} \frac{1}{h} \cdot \ln\left(\frac{3-2x-2h}{3-2x}\right)$$

$$= \lim_{h \to 0} \frac{1}{h} \cdot \ln\left(1 - \frac{2h}{3-2x}\right) = \lim_{h \to 0} \ln\left(1 + \frac{-2h}{3-2x}\right)^{\frac{1}{h}} =$$

$$= \ln \lim_{h \to 0} \left(1 + \frac{-2h}{3-2x}\right)^{\frac{1}{h}} = \ln \underbrace{\lim_{h \to 0} \left(1 + \frac{-2h}{3-2x}\right)^{\frac{3-2x}{-2h}}}_{e}{}^{\frac{-2h}{3-2x} \cdot \frac{1}{h}}$$

$$= \ln e^{\lim_{h \to 0} -\frac{2}{3-2x}} = -\frac{2}{3-2x}$$

$$\Rightarrow \boxed{f'(x) = -\frac{2}{3-2x}}$$

(e) $f(x) = \dfrac{1}{\sqrt{x-2}}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\dfrac{1}{\sqrt{x+h-2}} - \dfrac{1}{\sqrt{x-2}}}{h} =$$

$$= \lim_{h\to 0} \frac{\dfrac{\sqrt{x-2} - \sqrt{x+h-2}}{\sqrt{x+h-2}\cdot\sqrt{x-2}}}{h} = \lim_{h\to 0} \frac{\sqrt{x-2} - \sqrt{x+h-2}}{\sqrt{x+h-2}\cdot\sqrt{x-2}} \cdot \frac{1}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{x-2} - \sqrt{x+h-2}}{\sqrt{x+h-2}\cdot\sqrt{x-2}} \cdot \frac{\sqrt{x-2} + \sqrt{x+h-2}}{\sqrt{x-2} + \sqrt{x+h-2}} \cdot \frac{1}{h}$$

$$= \lim_{h\to 0} \frac{x-2-(x+h-2)}{h} \cdot \frac{1}{(\sqrt{x-2} + \sqrt{x+h-2})} \cdot \frac{1}{\sqrt{x+h-2}\cdot\sqrt{x-2}}$$

$$= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x-2} + \sqrt{x+h-2}),\sqrt{x+h-2}\cdot\sqrt{x-2}}$$

$$= \frac{1}{(\sqrt{x-2} + \sqrt{x-2})\cdot(\sqrt{x-2}\cdot\sqrt{x-2})} = \frac{1}{2(x-2)\sqrt{x-2}}$$

$$\Rightarrow \boxed{f'(x) = \frac{1}{2(x-2)\sqrt{x-2}}}$$

03) $s(t) = 5t^2$

$$v(t) = s'(t) = \lim_{h \to 0} \frac{s(t+h) - s(t)}{h} =$$

$$= \lim_{h \to 0} \frac{5(t+h)^2 - 5t^2}{h} = \lim_{h \to 0} \frac{\cancel{5t^2} - 10th + 5h^2 - 5t^2}{h}$$

$$= -10t \quad \Rightarrow \boxed{v(t) = -10t}.$$

[Obs: Nestes exercícios de apl. na Física pode-se derivar usando as regras de derivação. Neste exercício ainda fiz por definição, mas para os demais usarei as regras.]

Assim: $v(3) = -10 \cdot (3) = -30\,m/s$

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04) $s(t) = 100t^2 + 100t$

$s(t) = 39 \Rightarrow v(t) = ?$

$s(t) = 39 \Leftrightarrow 100t^2 + 100t = 39$

$\Leftrightarrow 100t^2 + 100t - 39 = 0$

$\Leftrightarrow t = \dfrac{-100 \pm \sqrt{10000 + 15600}}{200} = \dfrac{-100 \pm 160}{200}$

$\Leftrightarrow t = \dfrac{60}{200} = \dfrac{3}{10}\,s$

$v(t) = 200t + 100$

$v\left(\frac{3}{10}\right) = 200 \cdot \frac{3}{10} + 100 = 160\,m/s.$

05) $s(t) = t^{-2}$

Achar $v(t)$ para $t = 2s$ e $t = 3s$.

$$v(t) = s'(t) = -2t^{-3} = -\frac{2}{t^3}$$

Assim; $v(2s) = \frac{-\cancel{2}}{(2)^{\cancel{3}}} = -\frac{1}{4}$ m/s

$$v(3s) = -\frac{2}{(3)^3} = -\frac{2}{27} \text{ m/s.}$$

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06) $v_0 = v(0) = 160$ pés/s.

$$s(t) = 160t - 16t^2$$

$$v(t) = s'(t) = 160 - 32t.$$

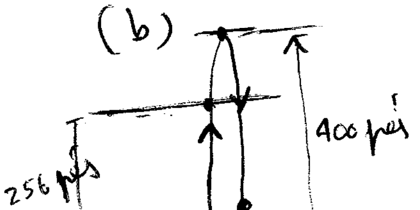
(a) A altura é máxima quando $v(t) = 0$. Assim, determinaremos o instante em que a altura foi máxima:

$$160 - 32t = 0 \iff t = \frac{160}{32} = 5s$$

Assim:

$$s(5s) = 160 \cdot (5) - 16 \cdot (5)^2 =$$
$$= 800 - 400 = 400 \text{ pés.}$$

(b)



$$s(t) = 256 = 160t - 16t^2 \iff$$
$$16t^2 - 160t + 256 = 0 \iff$$
$$t^2 - 10t + 16 = 0 \iff$$
$$t = \frac{10 \pm \sqrt{100 - 64}}{2} = \frac{10 \pm 6}{2}$$
$$t = 8s \text{ (desc.)} \quad \text{e } t = 2s \text{ (sub.)}$$

segue ... ⟹

Assim, $v(8s) = 160 - 32.(8) = -96$ pés/s
(veloc. na descida)

$v(2s) = 160 - 32.(2) = 96$ pés/s
(veloc. na subida)

Portanto, $|v(t)| = 96$, quando o objeto está a 256 pés do solo.

(c) Pelo item (a) vimos que o objeto leva 5s para atingir a altura máxima. Levará mais 5s para, a partir daí, atingir o solo, totalizando 10s.

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08) $P = \left(\frac{40+T}{140}\right)^5$, $T > 80$.

taxa instantânea: $P'(T) = \frac{dP}{dT}$.

Assim $\frac{dP}{dT} = 5 \cdot \left(\frac{40+T}{140}\right)^4 \cdot \frac{1}{140} = \frac{1}{28} \cdot \left(\frac{40+T}{140}\right)^4$

NOTE QUE USAMOS A REGRA $(v^k)' = k v^{k-1} \cdot v'$, COM $v = \frac{1}{140} \cdot (40+T)$

Logo, $P'(100) = \frac{1}{28} \cdot \left(\frac{40+100}{140}\right)^4 = \frac{1}{28}$ atm.

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09) $q(t) = -\frac{A}{w} \cdot \cos(w \cdot t + \varnothing)$

$$i = \frac{dq}{dt} = q'(t).$$

$$\Rightarrow i = -\frac{A}{w} \cdot (-\text{sen}(wt + \varnothing) \cdot w)$$

$$\Rightarrow i = +\frac{A}{\cancel{w}} \cdot \cancel{w} \cdot \text{sen}(wt + \varnothing)$$

$$\Rightarrow \boxed{i = A \cdot \text{sen}(w \cdot t + \varnothing)}$$

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11) (g) $f(x) = \frac{1 + x^{\frac{1}{2}}}{1 - x^{\frac{1}{2}}} = \frac{u}{v}$

$$\begin{cases} u = 1 + x^{\frac{1}{2}} \Rightarrow u' = \frac{1}{2} \cdot x^{-\frac{1}{2}} \cdot 1 = \frac{1}{2\sqrt{x}} \\ v = 1 - x^{\frac{1}{2}} \Rightarrow v' = -\frac{1}{2} \cdot x^{-\frac{1}{2}} \cdot 1 = -\frac{1}{2\sqrt{x}} \end{cases}$$

Assim;

$$f'(x) = \frac{v \cdot u' - u \cdot v'}{v^2} = \frac{(1-\sqrt{x}) \cdot \frac{1}{2\sqrt{x}} - (1+\sqrt{x}) \cdot \left(-\frac{1}{2\sqrt{x}}\right)}{(1-\sqrt{x})^2}$$

$$= \frac{\frac{1 - \cancel{\sqrt{x}} + 1 + \cancel{\sqrt{x}}}{2\sqrt{x}}}{(1-\sqrt{x})^2} = \frac{1}{\sqrt{x}} \cdot \frac{1}{(1-\sqrt{x})^2}$$

(j) $f(x) = 2x + \ln(\underbrace{\cos(1-2x)}_{v})$

$$f'(x) = 2 + \frac{v'}{v}$$

$$f'(x) = 2 + \frac{2\,\text{sen}(1-2x)}{\cos(1-2x)}$$

$$\Rightarrow \boxed{f'(x) = 2 + 2.\tan(1-2x)}$$

$$\left\{\begin{aligned} v &= \cos(1-2x) \\ v' &= -\text{sen}(1-2x).(-2) \\ &= +2\,\text{sen}(1-2x) \end{aligned}\right.$$

(l) $f(x) = \dfrac{\text{sen}\sqrt{x}}{1-\tan\sqrt{x}} = \dfrac{u}{v}$

$$\left[\begin{aligned} u &= \text{sen}\, x^{1/2} \Rightarrow u' = \cos x^{\frac{1}{2}}.\ \frac{1}{2}x^{-\frac{1}{2}}.1 = \frac{\cos\sqrt{x}}{2\sqrt{x}} \\ v &= 1-\tan\sqrt{x} \Rightarrow v' = -\sec^2\sqrt{x}\ .\ \frac{1}{2}x^{-\frac{1}{2}}.1 = \frac{-\sec^2\sqrt{x}}{2\sqrt{x}} \end{aligned}\right.$$

Logo;

$$f'(x) = \frac{v.u' - u.v'}{v^2} = \frac{(1-\tan\sqrt{x}).\frac{\cos\sqrt{x}}{2\sqrt{x}} - \text{sen}\sqrt{x}.\left(-\frac{\sec^2\sqrt{x}}{2\sqrt{x}}\right)}{(1-\tan\sqrt{x})^2}$$

$$= \frac{\frac{\cos\sqrt{x}}{2\sqrt{x}} - \tan\sqrt{x}.\frac{\cos\sqrt{x}}{2\sqrt{x}} + \frac{\text{sen}\sqrt{x}.\sec\sqrt{x}}{2\sqrt{x}}}{(1-\tan\sqrt{x})^2}$$

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12) $f(x) = e^{-x}$ $\qquad f'(x) = -e^{-x}$

$f(0) + x \cdot f'(0) = ?$ $\qquad$ Obs.: $[(e^{v})' = e^{v} \cdot v']$

Assim; $f(0) = e^{0} = 1$ e

$f'(0) = -e^{0} = -1$, e então:

$f(0) + x \cdot f'(0) = 1 + x \cdot (-1) = \underline{\underline{1 - x}}$

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15) $y = x \cdot e^{-x} = u \cdot v$ $\qquad \left[\begin{array}{l} u = x \Rightarrow u' = 1 \\ v = e^{-x} \Rightarrow v' = -e^{-x}. \end{array} \right.$

$\Rightarrow y' = u \cdot v' + u' \cdot v$

$\Rightarrow y' = x \cdot (-e^{-x}) + 1 \cdot e^{-x} = e^{-x} - x \cdot e^{-x}$

$\Rightarrow y' = e^{-x} \cdot (1 - x)$.

Logo;

$x \cdot y' = \underbrace{x \cdot e^{-x}}_{y}(1-x) = y \cdot (1-x)$. $\square$
