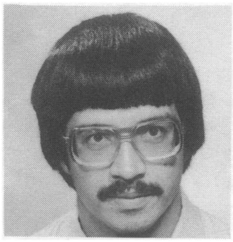
The Remainder in Taylor's Formula*

ESTEBAN I. POFFALD, *Wabash College, Crawfordsville, IN*

ESTEBAN I. POFFALD completed his undergraduate studies at Universidad Catolica de Chile, Temuco, Chile. He received an M.S. in Mathematics from Universidad Tecnica de Chile for work done under the direction of G. Riera. His Ph.D. dissertation was done under the supervision of S. Reich at the University of Southern California. He has been at Wabash since 1985.



We present here a mean-value theorem that generalizes the Taylor-Lagrange formula. The result arises in a natural way when one studies the asymptotic behavior of the remainder term of the formula. As an *application* of our result, we derive several numerical schemes to approximate the solution to initial-valued first order differential equations.

To begin, let us recall the Taylor-Lagrange formula. For convenience, we work in intervals of the form $[0, x]$.

THEOREM. (Taylor-Lagrange formula). *If f is continuous in $[0, x]$, $f^{(n-1)}(0)$ exists and $f^{(n)}(t)$ exists in $(0, x)$, then there exists a ξ in $(0, x)$ such that*

$$f(x) = p_{n-1}(x) + f^{(n)}(\xi)\frac{x^n}{n!}, \tag{1}$$

where

$$p_{n-1}(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + \cdots + f^{(n-1)}(0)\frac{x^{n-1}}{(n-1)!}$$

is the Taylor polynomial of order $n-1$ for f about 0.

Our first result concerns the asymptotic behavior as $x \to 0^+$ of the number ξ in the theorem above:

THEOREM 1. *With notation as in the previous theorem, if $f^{(n+1)}(t)$ exists in $[0, x]$, is continuous from the right at $t = 0$ and if $f^{(n+1)}(0) \neq 0$, then*

$$\lim_{x \to 0+} \frac{\xi}{x} = \frac{1}{n+1}.$$

Remark. The assumptions in this theorem imply that the number ξ is uniquely determined for x small enough.

*This work was supported in part by the Byron K. Trippet Research Stipend at Wabash College.

Proof of the theorem. In (1), we apply the Mean-Value theorem to $f^{(n)}(\xi)$: there is a number τ in $(0, \xi)$ such that

$$\begin{aligned} f(x) &= p_{n-1}(x) + f^{(n)}(\xi)\frac{x^n}{n!} \\ &= p_{n-1}(x) + \left[f^{(n)}(0) + f^{(n+1)}(\tau)\xi\right]\frac{x^n}{n!} \\ &= p_n(x) + f^{(n+1)}(\tau)\xi\frac{x^n}{n!}. \end{aligned}$$

On the other hand, by the Taylor-Lagrange formula

$$f(x) = p_n(x) + f^{(n)}(\sigma)\frac{x^{n+1}}{(n+1)!} \quad \text{for some } \sigma \text{ in } (0, x).$$

Therefore,

$$f^{(n)}(\tau)\xi = f^{(n)}(\sigma)\frac{x}{n+1}.$$

Since $f^{(n+1)}$ is right-continuous and nonzero at $t = 0$, the conclusion follows.

With notation as above, $f^{(n)}(\xi)\frac{x^n}{n!}$ can be viewed as the error made in approximating $f(x)$ by $p_{n-1}(x)$. In view of Theorem 1, one would expect that replacing ξ by $x/(n+1)$ and approximating $f(x)$ by

$$A_0(x) = p_{n-1}(x) + f^{(n)}\left(\frac{x}{n+1}\right)\frac{x^n}{n!}$$

would result in an approximation to $f(x)$ of order at least that of $p_n(x)$. In fact, it turns out that this approximation is of the same order as $p_{n+1}(x)$. We no longer require that $f^{(n+1)}(0) \neq 0$.

THEOREM 2. *If $f^{(n+2)}(t)$ exists and is continuous in $[0, x]$, then there exists a ξ in $(0, x)$ such that*

$$\begin{aligned} f(x) &= A_0(x) + \frac{n}{2(n+1)}f^{(n+2)}(\xi)\frac{x^{n+2}}{(n+2)!} \\ &= p_{n-1}(x) + f^{(n)}\left(\frac{x}{n+1}\right)\frac{x^n}{n!} + \frac{n}{2(n+1)}f^{(n+2)}(\xi)\frac{x^{n+2}}{(n+2)!}. \end{aligned}$$

Proof. Using Taylor's formula with integral remainder, we can write

$$f^{(n)}\left(\frac{x}{n+1}\right) = f^{(n)}(0) + f^{(n+1)}(0)\frac{x}{n+1} + \int_0^{\frac{x}{n+1}} f^{(n+2)}(t)\left[\frac{x}{n+1} - t\right]dt.$$

Therefore,

$$A_0(x) = p_{n+1}(x) + \frac{x^n}{n!}\int_0^{\frac{x}{n+1}} f^{(n+2)}(t)\left[\frac{x}{n+1} - t\right]dt$$

but

$$f(x) = p_{n+1}(x) + \frac{1}{(n+1)!}\int_0^x f^{(n+2)}(t)(x-t)^{n+1}\,dt.$$

It follows that

$$\begin{aligned} f(x) - A_0(x) &= \frac{1}{(n+1)!}\int_0^x f^{(n+2)}(t)(x-t)^{n+1}\,dt \\ &\quad - \frac{x^n}{n!}\int_0^{\frac{x}{n+1}} f^{(n+2)}(t)\left[\frac{x}{n+1} - t\right]dt \\ &= \int_0^{\frac{x}{n+1}} f^{(n+2)}(t)\left[\frac{(x-t)^{n+1}}{(n+1)!} - \frac{x^n}{n!}\left[\frac{x}{n+1} - t\right]\right]dt \\ &\quad + \frac{1}{(n+1)!}\int_{\frac{x}{n+1}}^x f^{(n+2)}(t)(x-t)^{n+1}\,dt. \end{aligned}$$

Now, let

$$g(t) = \frac{(x-t)^{n+1}}{n+1} - x^n\left(\frac{x}{n+1} - t\right).$$

Then $g(0) = 0$ and $g'(t) > 0$ in $(0, x]$, so that $g(t) \geqslant 0$ in that interval. Therefore, we can apply the Mean-Value theorem for integrals to deduce that there are numbers ξ_1 and ξ_2 in $(0, x)$ such that

$$\begin{aligned} f(x) - A_0(x) &= \frac{1}{n!} f^{(n+2)}(\xi_1)\int_0^{\frac{x}{n+1}} g(t)\,dt \\ &\quad + \frac{1}{(n+1)!} f^{(n+2)}(\xi_2)\int_{\frac{x}{n+1}}^x (x-t)^{n+1}\,dt. \end{aligned}$$

Finally, an application of the Intermediate Value Theorem concludes the proof.

In view of the way in which the statement of Theorem 2 follows naturally from Theorem 1, it should be clear at this point that we should try to determine the asymptotic behavior as $x \to 0^+$ of the number ξ in Theorem 2 and then establish a result similar to the one just presented. We are thus led to consider approximations to $f(x)$ of the form

$$\begin{aligned} A_k(x) &= p_{n-1}(x) + M_0 f^{(n)}(c_0 x)\frac{x^n}{n!} + M_2 f^{(n+2)}(c_2 x)\frac{x^{n+2}}{(n+2)!} \\ &\quad + \cdots + M_{2k} f^{(n+2k)}(c_{2k} x)\frac{x^{n+2k}}{(n+2k)!}, \end{aligned}$$

where for a given n, the coefficients M_{2i} and c_{2i} depend only on i. Clearly, we have $M_0 = 1$ and $c_0 = \dfrac{1}{n+1}$.

In the spirit of Theorem 2, we want $A_k(x)$ to be an approximation to $f(x)$ of order $n + 2k + 2$. By expanding $f^{(n+2j)}(c_{2j}x)$ in Taylor polynomials about 0 and collecting terms, we see that the sequences $\{M_{2j}\}$ and $\{c_{2j}\}$ can be obtained recursively from

$$\frac{M_0}{n!}\frac{c_0^{2k}}{(2k)!}+\frac{M_2}{(n+2)!}\frac{c_2^{2k-2}}{(2k-2)!}+\cdots+\frac{M_{2k}}{(n+2k)!}=\frac{1}{(n+2k)!} \tag{2}$$

$$\frac{M_0}{n!}\frac{c_0^{2k+1}}{(2k+1)!}+\frac{M_2}{(n+2)!}\frac{c_2^{2k-1}}{(2k-1)!}+\cdots+\frac{M_{2k}}{(n+2k)!}\frac{c_{2k}}{1!}=\frac{1}{(n+2k+1)!}. \tag{3}$$

Then, one verifies readily that if $f^{(n+2k+2)}$ exists and is continuous at 0,

$$\lim_{x\to 0}\frac{f(x)-A_k(x)}{x^{n+2k+2}}=\frac{M_{2k+2}}{(n+2k+2)!}f^{(n+2k+2)}(0),$$

and, therefore, $A_k(x)$ is indeed an approximation to $f(x)$ of order $n+2k+2$. In fact, the following mean-value theorem holds.

THEOREM 3. *If $f^{(n+2k)}(t)$ exists and is continuous in $[0, x]$, then there exists a number ξ in $(0, x)$ such that*

$$f(x)=A_{k-1}(x)+M_{2k}f^{(n+2k)}(\xi)\frac{x^{n+2k}}{(n+2k)!}. \tag{4}$$

Moreover, $0<M_{2j}<1$ for $j\geqslant 1$ and $0<c_{2j}<1$ for $j\geqslant 0$.

Proof. By induction on k. For $k=1$, this is just Theorem 2. Let us assume that the result holds for $k>1$. Assume also that $0<M_{2j}<1$ for $j=1,\ldots,k$ and $0<c_{2j}<1$ for $j=0,\ldots,k-1$. Finally, as part of the induction hypothesis, assume that $g_{k-1}(\lambda)>0$ for $0<\lambda<1$, where

$$g_k(\lambda)=\frac{(1-\lambda)^{n+2k+1}}{(n+2k+1)!}-\frac{M_0(c_0-\lambda)_+^{2k+1}}{n!(2k+1)!}-\cdots-\frac{M_{2k}(c_{2k}-\lambda)_+}{(n+2k)!1!}.$$

As usual, z_+ denotes the largest of 0 and z, for any real number z. (That $g_0(\lambda)>0$ for $0<\lambda<1$ follows from Bernoulli's inequality.) First of all, observe that if f has $n+2k+1$ continuous derivatives and $f^{(n+2k+1)}(0)\neq 0$, then $\lim_{x\to 0+}(\xi/x)=c_{2k}$, where ξ in $(0, x)$ satisfies

$$f(x)=A_{k-1}(x)+\frac{M_{2k}}{(n+2k)!}f^{(n+2k)}(\xi).$$

Therefore, $0\leqslant c_{2k}\leqslant 1$. This, together with the induction hypotheses, implies that $g_k(0)=g_k(1)=0$. Notice also that $g_k''(\lambda)$ exists for $\lambda\neq c_{2k}$ and $g_k''(\lambda)=g_{k-1}(\lambda)$. This implies that g_k is convex in $[0, c_{2k}]$ and in $[c_{2k}, 1]$.

If $c_{2k}\neq 0,1$, then $g_k''(\lambda)>0$ for $\lambda\neq 0, c_{2k}, 1$ implies that g_k' is increasing in $[0, c_{2k}]$ and in $[c_{2k}, 1]$. We also have that $g_k'(0)=g_k'(1)=0$, which implies that g_k is convex and increasing in $[0, c_{2k}]$ and it is convex and decreasing in $[c_{2k}, 1]$. The conclusion is that $g_k(\lambda)\geqslant 0$ in $[0,1]$ with equality only for $\lambda=0$ and 1.

If c_{2k} were 0 or 1, then we could carry out the same analysis as above, using one-sided derivatives instead. But $c_{2k}=0$ or 1 would imply that $g_k\equiv 0$ in $[0,1]$, which would force $g_{k-1}\equiv 0$ in $[0,1]$, contradicting the induction hypotheses.

Now we are ready to complete the proof:
If f has $n + 2k + 2$ continuous derivatives in $[0, x]$, we have

$$f(x) - A_k(x) = f(x) - \sum_{j=0}^{k} M_{2j} f^{(n+2j)}(c_{2j}x)\frac{x^{n+2j}}{(n+2j)!}$$

Now, expand each term in the right-hand side into its Taylor polynomial of degree $2k - 2j + 1$ about 0, with integral remainder, and collect terms. One gets

$$f(x) - A_k(x) = \int_0^x f^{(n+2k+2)}(t)\frac{(x-t)^{n+2k+1}}{(n+2k+1)!}\,dt$$

$$- \sum_{j=0}^{x} \int_0^{c_{2j}x} M_{2j} f^{(n+2k+2)}(t)\frac{(c_{2j}x - t)^{2k-2j+1}}{(2k-2j+1)!}\frac{x^{n+2j}}{(n+2j)!}\,dt.$$

In each of the integrals above, substitute $t = \lambda x$. After rearranging, this gives

$$f(x) - A_k(x) = x^{n+2k+2}\int_0^1 f^{(n+2k+2)}(\lambda x) g_k(\lambda)\,d\lambda.$$

Since $g_k(\lambda) \geqslant 0$ in $[0,1]$, we can apply the Mean-Value theorem for integrals to conclude the proof.

Remarks.

1. *From the point of view of applications*, it would be interesting to have a closed formula for the sequences $\{c_{2j}\}$ and $\{M_{2j}\}$ given by (2) and (3). *Numerical computations* suggest that for fixed n the sequence $\{c_{2j}\}$ is monotonically increasing while $\{M_{2j}\}$ decreases to 0.

2. If f is analytic in a disk with center at the origin and radius $R > 0$, then one can use the Cauchy-Hadamard estimates to deduce that the expansion

$$f(x) = p_{n-1}(x) + \sum_{j=0}^{\infty} M_{2j} f^{(n+2j)}(c_{2j}x)\frac{x^{n+2j}}{(n+2j)!}$$

is valid in the disk $|x| < R/2$. In practice, however, the region of convergence of this expansion appears to be at least as large as that of the power series expansion of f. (*Cf. Remark* 4 *below*.) The validity of this statement may depend on the behavior of the sequences M_{2i} and c_{2i} as $i \to \infty$.

3. For $n = 1$, both the statement and proof of our main result are simpler: elementary combinatorial identities reveal that in this case $c_{2j} = 1/2$ and $M_{2j} = 1/4^j$ satisfy (2) and (3). This suggests the following proof for $n = 1$: In the Taylor-Lagrange formula

$$f(b) = f(a) + f'(a)(b-a) + \cdots + f^{(m)}(a)\frac{(b-a)^m}{m!} + f^{(m+1)}(\xi)\frac{(b-a)^{m+1}}{(m+1)!},$$

take $m = 2k$. Put $a = x/2$ and $b = x$; then, put $a = x/2$ and $b = 0$, subtract and appeal to the Intermediate Value Theorem, to deduce that there is a ξ in $(0, x)$ such that

$$f(x)=f(0)+\sum_{j=0}^{2k}\frac{2}{(2j+1)!}f^{(2j+1)}\left(\frac{x}{2}\right)\left[\frac{x}{2}\right]^{2j+1}$$
$$+\frac{2}{(2k+1)!}f^{(2k+1)}(\xi)\left[\frac{x}{2}\right]^{2k+1}.$$

It follows that if

$$\lim_{k\to\infty}\frac{f^{(2k+1)}(\xi)}{(2k+1)!}\left[\frac{x}{2}\right]^{2k+1}=0,$$

then

$$f(x)=f(0)+2\sum_{j=0}^{\infty}\frac{1}{(2j+1)!}f^{(2j+1)}\left(\frac{x}{2}\right)\left[\frac{x}{2}\right]^{2j+1}. \tag{5}$$

4. We are grateful to W. Swift for calling our attention to the fact that if f is analytic at 0, then (5) is valid for all x in the interior of the Borel polygon of summability of f. Recall that the Borel polygon of f is constructed as follows: Let P be a point in the complex plane where f has a singularity and let O denote the origin. Draw the line Δ_P perpendicular to $\overline{OP}$ at P. Then, a point x is in the Borel polygon of summability if and only if x is in the half plane determined by Δ_P and 0, for all the singular points P of f. It then follows that x is in the interior of the Borel polygon if and only if f is analytic at every point of the closed disk with diameter $\overline{Ox}$. Therefore, if x belongs to the Borel polygon of f, then

$$f(z)=\sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}\left(\frac{x}{2}\right)\left[z-\frac{x}{2}\right]^{n}$$

holds for all z such that $|z-(x/2)|\leqslant|x/2|$. As above, putting $z=x$ and then $z=0$ and subtracting gives the result.

Example. If $f(x)=\ln(1+x)$, then

$$f^{(2k+1)}(x)=\frac{(2k)!}{(1+x)^{2k+1}}.$$

Applying (5), we obtain

$$\ln(1+x)=2\sum_{n=0}^{\infty}\frac{1}{2n+1}\left[\frac{x}{2+x}\right]^{2n+1}.$$

The Borel polygon of f is the half plane $\operatorname{Re}(x)>-1$. Therefore, this expansion is valid for all x with $\operatorname{Re}(x)>-1$. This should be contrasted with the circle of convergence of the Taylor series expansion for $f(x)$ about $x=0$. Even in the common region of convergence, this expansion converges faster. Since

$$\arctan x=\frac{1}{2i}[\ln(1+ix)-\ln(1-ix)],$$

we also obtain

$$\arctan x=-i\sum_{n=0}^{\infty}\frac{1}{2n+1}\left[(x+2i)^{2n+1}-(x-2i)^{2n+1}\right]\left[\frac{x}{4+x^2}\right]^{2n+1},$$

which is valid for $-1 < \mathrm{Im}(x) < 1$. In particular, for all x real, $\arctan x$ may be computed using

$$\arctan x = \sum_{n=0}^{\infty} \frac{1}{2n+1} A_n(x)$$

where

$$A_0(x) = \frac{4x}{4+x^2}, \qquad A_1(x) = 4\left[\frac{x}{4+x^2}\right]^3 (3x^2 - 4)$$

and

$$A_{n+2}(x) = \left[\frac{x}{4+x^2}\right]^2 \left[2(x^2-4)A_{n+1}(x) - x^2 A_n(x)\right].$$

For example,

$$\arctan 2 = \sum_{k=0} \frac{(-1)^k}{4^k}\left[\frac{1}{4k+1} + \frac{1}{8k+6}\right]$$

and

$$\pi = \frac{16}{5} \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{1}{2n+1} \binom{2n+1}{2j+1} \frac{(-4)^j}{5^{2n}}.$$

An Application. We will now use Theorem 3 to derive several numerical methods of approximating the solution of the initial-valued differential equation

$$y' = f(x, y) \qquad y(x_0) = y_0,$$

where f (and, therefore, y) is a sufficiently smooth function.

Fix $h > 0$. For each nonnegative integer j, we are seeking an approximation y_{j+1} to the exact value of the solution y at $x_{j+1} = x_0 + (j+1)h$. Let $f_j = f(x_j, y_j)$ and denote by E the error $y(x_{j+1}) - y_{j+1}$, assuming that $y(x_i) = y_i$ for $0 \leqslant i \leqslant j$ (E is the local discretization error). We will omit the derivation of the error term. Note, however, that in each case the error term can be derived using the main ideas in this note, namely expansion in Taylor polynomials with integral remainder followed by an application of the Mean-Value Theorem for integrals.

(i). The Modified Euler's Method: for $n = 1$ and $k = 1$, we apply (4) (in the interval $[x_{j-1}, x_{j+1}]$) to obtain the approximation $y_{j+1} = y_{j-1} + 2hf_j$ with $E = y'''(\xi)(h^3/3)$.

(ii). For $n = 2$ and $k = 1$ and the interval $[x_{j-2}, x_{j+1}]$, we get

$$y(x_{j+1}) \approx y(x_{j-2}) + 3hy'(x_{j-2}) + \frac{9h^2}{2} y''(x_{j-1}).$$

Now, using the approximation $y''(x_{j-1}) \approx [y'(x_j) - y'(x_{j-2})]/2h$ and rearranging, we get $y_{j+1} = y_{j-2} + (3h/4)[f_{j-2} + 3f_j]$ with the error term $E = (3/4)y^{(4)}(\xi)h^4$.

This approximation does not appear to be one of the standard multistep methods of elementary numerical analysis. However, applying it to approximate $\int_a^b g(t)\,dt$, we obtain

$$\int_a^b g(t)\,dt \approx \frac{1}{4}(b-a)\left[g(a)+3g\left(\frac{a+2b}{3}\right)\right].$$

This is clearly asymmetric with respect to a and b, which suggests also considering the approximation

$$\int_a^b g(t)\,dt \approx \frac{1}{4}(b-a)\left[g(b)+3g\left(\frac{b+2a}{3}\right)\right]$$

and then averaging these to obtain

$$\int_a^b g(t)\,dt \approx \frac{1}{8}(b-a)\left[g(a)+3g\left(\frac{a+2b}{3}\right)+3g\left(\frac{2a+b}{3}\right)+g(b)\right],$$

which is the well-known Simpson's 3/8 Rule, with error $-(1/6480)(b-a)^5g^{(4)}(\xi)$.

(iii). In (ii) above, we could equally well use the approximation $y''(x_{j-1}) \approx [y(x_j) - 2y(x_{j-1}) + y(x_{j-2})]/h^2$, which gives

$$y_{j+1} = \frac{1}{2}\left[11y_{j-2} - 18y_{j-1} + 9y_j\right] + 3hf_{j-2},$$

with $E = (3/4)y^{(4)}(\xi)h^4$.

(iv). Now, we use (4) with $n = 1$, $k = 2$ and the interval $[x_{j-3}, x_{j+1}]$ to get

$$y(x_{j+1}) \approx y(x_{j-3}) + y'(x_{j-1})4h + y'''(x_{j-1})\frac{(4h)^3}{24}.$$

Approximating $y'''(x_{j-1})$ as we approximated $y''(x_{j-1})$ in (iii) above, we derive the multistep method

$$y_{j+1} = y_{j-3} + \frac{4}{3}h\left[2f_{j-2} - f_{j-1} + 2f_j\right],$$

with the error $E = (14/45)y^{(5)}(\xi)h^5$.

(v). Again, we take $n = 1$ and $k = 2$, but we use the interval $[x_{j-1}, x_{j+1}]$. This gives $y(x_{j+1}) \approx y(x_{j-1}) + y'(x_j)2h + y'''(x_j)(2h)^3/24$. Using $y'''(x_j) \approx [y'(x_{j+1}) - 2y'(x_j) + y'(x_{j-1})]/h^2$, we obtain

$$y_{j+1} = y_{j-1} + (h/3)\left[f_{j+1} + 4f_j + f_{j-1}\right],$$

with $E = (-1/90)y^{(5)}(\xi)h^5$. Observe that this last formula, when interpreted as a numerical integration method, gives the well known Simpson's Rule. Also, notice that the formula is implicit, since y_{j+1} appears also in the right-hand side. However, using this formula in conjunction with the one derived in (iv), we obtain a well-known predictor-corrector method (Milne's method):

$$y_{j+1}^{(p)} = y_{j-3} + \frac{4h}{3}\left[2f_j - f_{j-1} + 2f_{j-2}\right]$$

$$y_{j+1} = y_{j-1} + \frac{h}{3}\left[f_{j+1}^{(p)} + 4f_j + f_{j-1}\right],$$

where $f_{j+1}^{(p)} = f(x_{j+1}, y_{j+1}^{(p)})$.

Conclusion. All the results above arose from studying the error term in the Taylor's polynomial approximation to a given function. By no means is this restricted to Taylor's polynomials: in general, suppose that $A(f)$ is a numerical approximation to a functional $F(f)$ such that $F(f) = A(f) + E(f)$ for all f in an appropriate class of functions. If anything can be said about the error term $E(f)$, then perhaps one should try to use $A(f) + \bar{E}(f)$ as a better approximation to $F(f)$. Here, $\bar{E}(f)$ is an "estimate" of the error that does not depend on the particular function f. To illustrate this, consider the trapezoidal rule in numerical integration:

$$\int_a^b f(t)\,dt \approx \frac{(b-a)}{2}[f(a)+f(b)].$$

Now, the error of this approximation is $-f''(\xi)(b-a)^3/12$, for some ξ in (a, b). It can then be shown that if $f^{(3)}$ is continuous and nonzero at a, then, as $b \to a^+$, c approaches the midpoint of $[a, b]$, that is,

$$\lim_{b\to a+} \frac{(c-a)}{(b-a)} = \frac{1}{2}.$$

Therefore, we obtain the numerical integration method:

$$\int_a^b f(t)\,dt \approx \frac{(b-a)}{2}[f(a)+f(b)] - f''\left(\frac{a+b}{2}\right)\frac{(b-a)^3}{12}$$

with error $-(1/480)f^{(4)}(\xi)(b-a)^5$. Finally, we note that if we approximate

$$f''\left(\frac{a+b}{2}\right) \quad \text{by} \quad \left[f(a) - 2f\left(\frac{a+b}{2}\right) + f(b)\right] \Big/ \frac{(b-a)^2}{2}$$

and replace, we obtain the approximation

$$\int_a^b f(t)\,dt \approx \frac{(b-a)}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right],$$

i.e., Simpson's rule once again.

REFERENCES

1. R. Ellis and D. Gulick, Calculus with Analytic Geometry, 3rd ed., H. B. J. Publishers, 1986.
2. J. M. H. Olmsted, Real Variables, Appleton-Century-Crofts, Inc., 1959.
3. S. D. Conte and C. de Boor, Elementary Numerical Analysis, 2nd ed., McGraw-Hill, 1972.
4. J. Ll. Morris, Computational Methods in Elementary Numerical Analysis, John Wiley & Sons, 1983.
5. E. Borel, Leçons sur les Séries Divergentes, Gauthier-Villars, 1901.