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ON THE DERIVATIVE OF A DISCONTINUOUS FUNCTION

BY

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In this note we shall deal with finite real functions defined on the interval $I = (0,1)$. For a given function f we shall denote by C_f the set of its continuity points, by D_f the set of its discontinuity points and by Δ_f^* the set of points at which f has a derivative (finite or infinite).

Kronrod [2] has proved that a necessary and sufficient condition for a set E to be the set of discontinuity points of a function f with a finite derivative at every continuity point is that $E \epsilon (F_\sigma \frown G_\delta)$, i. e., E is both an F_σ and a G_δ-set. As indicated by examples of functions with a derivative everywhere, given by Garg [1] and Marcus [4], for a larger class of functions f having a finite or infinite derivative at its continuity points the condition $E \epsilon G_\delta$ is not more necessary that $E = D_f$ for a function f of that class. The set of discontinuity points of the mentioned functions is dense and countable and thus it is not a G_δ-set. In this connection Garg [1] asked if for every set $E \epsilon F_\sigma$ there exists a function f such that $E = D_f$ and $\Delta_f^* \supset C_f$. The answer to this question is "no", as stated in corollary 1 below. It is worth while to note that as well the condition of $E \epsilon (F_\sigma \frown G_\delta)$ as the condition of E being countable are sufficient for E to be the set of discontinuity points of a function f of the class spoken of. This follows from a theorem of Marcus [4], according to which for every countable set there exists a function having a derivative everywhere and such that the countable set in question is the set of its discontinuity points. The reader may compare also a generalization of this theorem given by Lipiński [3].

We shall use in some proofs the notion of a point of asymmetrical structure of a function, introduced by Young. A number l with $-\infty \leqslant l \leqslant \infty$ will be called a *left-hand limiting value* of function f at point $x \epsilon I$, if there exists a sequence $x_1, x_2, x_3, \ldots$ of points from I such that $x_n < x$ for $n = 1, 2, \ldots, \lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} f(x_n) = l$. A *right-hand limiting value* is defined analogously. A number $x \in I$ will be called a *point of asymmetrical structure* of f, if there exists a number l which either is a left-hand limiting value of f and is not a right-hand limiting value of f at x, or it is a right-hand limiting value of f at x and is not a left-hand one.

Further notation: $\bar{f}(x)$ and $\underline{f}(x)$ are upper and lower derivatives of f at point x; E^c is the set of condensation points of the set E; A_f is the set of asymmetrical structure of function f; Δ_f is the set of points at which f has a finite derivative.

LEMMA. *Let f be an arbitrary function and a a number with $-\infty \leqslant a \leqslant \infty$. If any one of the sets $\{x|\bar{f}(x) \geqslant a\}$ and $\{x|\underline{f}(x) \leqslant a\}$ is dense on I, then it is residual on I.*

Proof. We shall provide the proof for the set $\{x|\bar{f}(x) \geqslant a\}$ only, for in the other case the reasoning is quite analogous.

We can assume that $a \neq \pm\infty$, since for $a = -\infty$ the lemma is evidently true and for $a = +\infty$ it is implied by the case of a finite a. In fact, if for every natural n the sets $\{x|\bar{f}(x) \geqslant n\}$ are residual in I, then so is the set

$$\{x|\bar{f}(x) \geqslant \infty\} = \bigcap_{n=1}^{\infty} \{x|\bar{f}(x) \geqslant n\}.$$

First we shall prove under an additional assumption of $Y \cap C_f$ being dense in every open interval $Y \subset I$ that the set $Y \cap C_f \cap \{x|\bar{f}(x) \geqslant a\}$ is residual on Y. To this end let $K \subset Y$ be a non-empty open interval and let $C_f = \bigcap_{n=1}^{\infty} G_n$, where G_n are open sets satisfying the condition $G_n \supset G_{n+1}$ for $n = 1, 2, 3, \ldots$ The sets $G_n \cap Y$ are dense on Y. Hence we conclude that there exists a non-empty open interval $K' \subset K \cap G_1$. In the interval K' there exist two points x_1 and y_1 such that

$$0 < y_1 - x_1 < 1, \qquad \frac{f(y_1)-f(x_1)}{y_1-x_1} > a-1 \quad \text{and} \quad \langle x_1, y_1\rangle \subset G_1.$$

For if such points did not exist, then for any two points $x, y \in K'$ we had

$$\frac{f(y)-f(x)}{y-x} \leqslant a-1,$$

and, consequently, we had $\bar{f}(x) \leqslant a-1$ for any $x \in K'$. This, however, yields a contradiction with the assumption of $\{x|\bar{f}(x) \geqslant a\}$ being dense.

Suppose we have defined, for $n > 1$, the points x_n and y_n satisfying the conditions

$$0 < y_n - x_n < \frac{1}{n}, \qquad \frac{f(y_n)-f(x_n)}{y_n - x_n} > a - \frac{1}{n},$$

$$x_{n-1} < x_n < y_n < y_{n-1} \quad \text{and} \quad \langle x_n, y_n\rangle \subset G_n.$$

In the interval (x_n, y_n) there exists a non-empty open interval K'' $\subset (x_n, y_n) \cap G_{n+1}$ of lenght smaller than $1/(n+1)$ and in K'' there are two points x_{n+1} and y_{n+1} such that

$$0 < y_{n+1} - x_{n+1} < \frac{1}{n+1}, \qquad \frac{f(y_{n+1})-f(x_{n+1})}{y_{n+1} - x_{n+1}} > a - \frac{1}{n+1},$$

$$x_n < x_{n+1} < y_{n+1} < y_n, \qquad \langle x_{n+1}, y_{n+1}\rangle \subset G_{n+1}.$$

For if no such points were in K'', we would get, as in the case of K', a contradiction with the assumption of $\{x|\bar{f}(x) \geqslant a\}$ being dense. The intersection of the intervals (x_n, y_n) is a one-point set $\langle \xi \rangle$.

For every n we have

$$a - \frac{1}{n} < \frac{f(y_n)-f(x_n)}{y_n - x_n} = \frac{f(y_n)-f(\xi)+f(\xi)-f(x_n)}{y_n - \xi + \xi - x_n}$$
$$\leqslant \max\left(\frac{f(y_n)-f(\xi)}{y_n - \xi}, \frac{f(\xi)-f(x_n)}{\xi - x_n}\right).$$

Let ξ_n be that of the points x_n and y_n for which we have

$$\frac{f(\xi_n)-f(\xi)}{\xi_n - \xi} = \max\left(\frac{(f(y_n)-f(\xi)}{y_n - \xi}, \frac{f(\xi)-f(x_n)}{\xi - x_n}\right).$$

We thus have

$$|\xi_n - \xi| < y_n - x_n < \frac{1}{n} \quad \text{and} \quad \frac{f(\xi_n)-f(\xi)}{\xi_n - \xi} > a - \frac{1}{n}$$

for every n. These inequalities imply $\bar{f}(\xi) \geqslant a$. Moreover, for every n we have $\xi \in (x_n, y_n) \subset K \cap G_n$ and thus $\xi \in K \cap C_f$. From this and the preceding sentence we conclude that $\xi \in K \cap C_f \cap \{x|\bar{f}(x) \geqslant a\}$.

We have thus proved that for any open interval $K \subset Y$ there exists a point $\xi \in K \cap C_f \cap \{x|\bar{f}(x) \leqslant a\}$, which means that the set $Y \cap C \cap \{x|\bar{f}(x) \geqslant a\}$ is dense on Y. Now, as proved by Zahorski [7], the set $Y \cap C_f \cap \{x|\bar{f}(x) \geqslant a\}$ is a G_δ-set. Consequently it is residual on Y.

Let us pass now to the proof of the lemma in its full generality. Suppose the set $\{x|\bar{f}(x) \geqslant a\}$ is not residual in I. The set $\{x|\bar{f}(x) < a\}$ is residual on a certain interval Y, because it is an $F_{\sigma\delta}$-set of the second category. We have $\bar{f}(x) = \infty$ at points $x \in D_f \setminus A_f$. Therefore the set $D_f \cap \{x|\bar{f}(x) < a\}$ is countable, as it is a part of a countable set A_f

(see [6]). We thus conclude that the set $Y \cap C_f \supset Y \cap C_f \cap \{x | \bar{f}(x) < a\}$ is dense on Y (it is even residual on Y). Hence and in view of the first part of the proof the set $Y \cap \{x | \bar{f}(x) \geqslant a\} \supset Y \cap C_f \cap \{x | \bar{f}(x) \geqslant a\}$ is residual on Y. We have got a contradiction, for the sets $Y \cap \{x | \bar{f}(x) < a\}$ and $Y \cap \{x | \bar{f}(x) \geqslant a\}$ cannot be residual simultaneously.

The proof of the lemma is complete.

THEOREM 1. *If the set of discontinuity points of a function f has the power of the continuum on every subinterval of I, then Δ_f^* is of the first category on I.*

Proof. The set A_f is countable. Consequently the set $D_f \setminus A_f$ has the power of the continuum on every interval contained in I. But for $x \in D_f \setminus A_f$ we have $\underline{f}(x) = -\infty$ and $\bar{f}(x) = \infty$. Hence the sets $\{x | \underline{f}(x) \leqslant -\infty\}$ and $\{x | \bar{f}(x) \geqslant \infty\}$ are dense, and, in view of lemma, they are residual on I. Since $\Delta_f^* \subset I \setminus (\{x | \underline{f}(x) \leqslant -\infty\} \cap \{x | \bar{f}(x) \geqslant \infty\})$, the set Δ_f^* is of the first category on I, q. e. d.

COROLLARY 1. *If an F_σ-set $E \subset I$ is of the first category on I and has the power of the continuum on every subinterval of I, then there is no function f such that $E = D_f$ and $\Delta_f^* \supset I \setminus E$.*

Proof. By virtue of the conditions imposed on E and of theorem 1 we infer that if $E = D_f$ for a function f, then Δ_f^* is of the first category and thus f cannot have a derivative $f'(x)$ at every point x of the residual set $I \setminus E$, q. e. d.

Remark. If $E \in F_\sigma$ and $|E| = 1$, then in view of theorem 1 there is no function f such that $D_f = E$ and the set $\{x | f'(x) = \infty\}$ is residual on I.

In fact, the equations $D_f = E$ and $E^c = \langle 0,1 \rangle$ imply that Δ_f^* is of the first category on I.

Marcus [5] has proved that for any number a with $0 \leqslant a \leqslant 1$ there exists a function φ with a dense set of discontinuity points of measure a such that it has on I a residual set of points at which its right-hand derivative exists and is equal to $+\infty$. It follows from the Remark that for $a = 1$ it is not possible to strengthen this theorem through replacing the right-hand derivative by derivative. However, theorem 2 shows that it is possible if $0 \leqslant a < 1$.

THEOREM 2. *Let A and B be two sets such that:*

(a) *$A \subset I$ and A is countable,*

(b) *$B \subset I$, $B \in F_\sigma$ and B is nowhere dense on I.*

Then there exists a function φ with the following properties:

(1) $D_\varphi = A \cup B$,

(2) *$\Delta_f^* \supset I \setminus \bar{B}$, where $\bar{B}$ is the closure of B,*

(3) *$\{x | \varphi'(x) = \infty\}$ is residual on I.*

Proof. The set $A \setminus B$ is countable. Arrange its points into a sequence $a_2, a_3, a_4, \ldots$ Put $a_1 = 0$ and $f(x) = \sum_{a_n < x} b_n$, where $b_1 = 0$ and $b_n = 2^{-n}$ for $n = 2, 3, 4, \ldots$ The function f is non-decreasing and $D_f = A \setminus B$. The set $I \setminus \Delta_f$ has measure 0. Therefore there exists a G_δ-set $C \subset I$, dense on I and such that $|C| = 0$ and $I \setminus \Delta_f \subset C$. By a theorem of Zahorski [8] there exists an increasing continuous function g defined on I such that $\Delta_g^* = I$ and $\{x | g'(x) = \infty\} = C$.

Let $B = \bigcup_{n=1}^{\infty} B_n$, where B_n are closed sets such that $B_n \subset B_{n+1}$ for $n = 1, 2, 3, \ldots$ The function

$$h(x) = \begin{cases} 0 \text{ if } x \in I \setminus B, \\ \dfrac{1}{n} \quad \text{if} \quad x \in B_n \setminus B_{n-1}, \end{cases}$$

where $n = 1, 2, 3, \ldots$ and B_0 is the empty set, is discontinuous at every point $x_0 \in B$ and it is continuous at the remaining points.

In fact, if $x_0 \in B$, then $h(x_0) > 0$ and in every neighbourhood of x_0 there is a point x of the set $I \setminus B$ at which $h(x) = 0$. Now, if $x_0 \in I \setminus B$, then we have $x_0 \in I \setminus B_n$ for every n. Let ε be an arbitrary positive number and n_0 a positive integer such that $1/n_0 < \varepsilon$. There exists a $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subset I \setminus B_{n_0}$. For $x \in (x_0 - \delta, x_0 + \delta)$ we have $h(x) \leqslant 1/n_0$ and $|h(x) - h(x_0)| = |h(x)| \leqslant 1/n_0 < \varepsilon$. This proves that the function h is continuous at points $x_0 \in I \setminus B$. Evidently $\Delta_h \supset \{x | h'(x) = 0\} \supset I \setminus \bar{B}$.

We now define the function searched for by

$$\varphi(x) = f(x) + g(x) + h(x).$$

As D_f and D_h are disjoint, we have $D_\varphi = D_f \cup D_h = (A \setminus B) \cup B = A \cup B$. In order to prove (2) and (3) let us note that if $x_0 \notin \bar{B} \cup C$, then the derivatives $f'(x_0)$, $g'(x_0)$ and $h'(x_0)$ exist and are finite, and if $x_0 \in C \setminus \bar{B}$, then there exists a $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subset I \setminus \bar{B}$. Now, if $x \in (x_0 - \delta, x_0 + \delta)$, then $h(x) = h(x_0) = 0$ and

$$\frac{\varphi(x) - \varphi(x_0)}{x_0 - x_0} = \frac{f(x) - f(x_0)}{x - x_0} + \frac{g(x) - g(x_0)}{x - x_0} + \frac{h(x) - h(x_0)}{x - x_0}$$

$$= \frac{f(x) - f(x_0)}{x - x_0} + \frac{g(x) - g(x_0)}{x - x_0} \geqslant \frac{g(x) - g(x_0)}{x - x_0},$$

because the function f is non-decreasing. The inequality and equation $g'(x_0) = \infty$ imply $\varphi'(x_0) = \infty$, q. e. d.

THEOREM 3. *If the set of points at which a function f has a (finite or infinite) derivative is residual on I, then $D_f = A \cup B$, where A is an F_σ-set nowhere dense on I and B is countable.*

Proof. We have $D_f = (D_f \cap D_f^c) \cup (D_f \setminus D_f^c) = A \cup B$. The set $B = D_f \setminus D_f^c$ is countable, where as $A = D_f \cap D_f^c$ is an F_σ-set, because D_f^c is closed. Suppose A is not a nowhere dense set. Then there exists an interval $Y \subset I$ on which the set $Y \cap A$ is dense. In every subinterval of Y there exists a point of the set $D_f^c \supset A$ and, consequently, the points of D_f belonging to Y form a set of the power of the continuum. Hence and from theorem 1 it follows that Δ_f^* is of the first category on Y. This, however, contradicts the assumption of the theorem. The proof is thus completed.

Theorems 2 and 3 characterize the set D_f of function f which have a derivative on a residual set. We can deduce from theorem 3 the theorem of Garg we have spoken about in the introduction by substituting $B = 0$.

THEOREM 4. *If the set of points at which a function f has no finite derivative is of positive measure on every subinterval of I, then Δ_f^* is of the first category on I.*

Proof. Let $Y \subset I$ be an arbitrary non-empty open interval. The function f is not of bounded variation on $\overline{Y}$, because $|Y \setminus \Delta_f| > 0$. Therefore the function $F(x) = f(x) - x$ is not decreasing on Y. Thus there exist points $x_1, y_1 \in Y$ such that $x_1 < y_1$ and $F(x_1) \leqslant F(y_1)$. Hence we get

$$\frac{f(y_1)-f(x_1)}{y_1-x_1} \geqslant 1.$$

Suppose we have defined, for $n > 1$, the points x_n and y_n such that

$$(*) \qquad x_{n-1} < x_n < y_n < y_{n-1}, \qquad y_n - x_n < \frac{1}{n},$$

$$\frac{f(y_n)-f(x_n)}{y_n-x_n} \geqslant n.$$

We prove in the same manner as above that in an interval $\langle x_n', y_n'\rangle \subset (x_n, y_n)$ of length smaller than $1/(n+1)$ there exist points x_{n+1} and y_{n+1} such that

$$x_n < x_{n+1} < y_{n+1} < y_n, \qquad y_{n+1} - x_{n+1} < \frac{1}{n+1},$$

$$\frac{f(y_{n+1})-f(x_{n+1})}{y_{n+1}-x_{n+1}} \geqslant 1+n.$$

In this way we define by induction the sequences $\{x_n\}$ and $\{y_n\}$ of points satisfying $(*)$.

Put $\langle\xi\rangle = \bigcap_{n=1}^{\infty} (x_n, y_n)$. We obviously have $\bar{f}(\xi) = \infty$. This means that the set $\{x \mid \bar{f}(x) = \infty\}$ is dense on I and in view of the lemma it is residual on I.

We prove analogously that also the set $\{x \mid \underline{f}(x) = -\infty\}$ is residual. Now Δ_f^* is a part of $I \setminus (\{x \mid \bar{f}(x) = \infty\} \cap \{x \mid \underline{f}(x) = -\infty\})$. Hence Δ_f^* is of the first category on I, q. e. d.

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