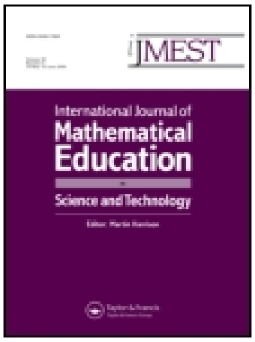

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On a discontinuity of a derivative

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$$l/b \leqslant \lim_{n\to\infty} n\sin(\pi/n) = \lim_{n\to\infty} 2n\sin\left(\tfrac{1}{2}\pi/n\right) = \pi$$

These are consistent with the previously found expressions in all four cases.

On a discontinuity of a derivative

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Many texts in 'advanced calculus' present Darboux's Theorem (also known as the Intermediate Value Theorem for Derivatives) and the well-known example

$$f(x) = \begin{cases} x^2 \sin\dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

of a function with discontinuous derivative at the origin. But these texts typically fail to discuss the relationship between Darboux's result and the type of discontinuity a given derivative must have at such a point. It is no accident, for example, that the discontinuity of

$$f'(x) = \begin{cases} 2x\sin\dfrac{1}{x} - \cos\dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

at the origin is such that $\lim_{x\to 0} f'(x)$ does not exist. In this paper, we precisely identify such discontinuities. The arguments stay within the realm of elementary classical analysis and are thus accessible to students encountering a first proof course in the subject.

1. Introduction

Let us begin with some appropriate notation, some definitions and a review of the statement of Darboux's Theorem. Let $\Re$ designate the set of real numbers and suppose $F\colon [a,b] \to \Re$. Let $x_0 \in [a,b]$. If there exists $L \in \Re$ such that

$$\lim_{x\to x_0} F(x) = L \quad \text{but } L \neq F(x_0)$$

x_0 is called a *removable discontinuity* of F.

If there exist $L_1, L_2 \in \Re$ such that

$$F(x_0^-) = \lim_{x\to x_0^-} F(x) = L_1 \quad \text{and} \quad F(x_0^+) = \lim_{x\to x_0^+} F(x) = L_2 \quad \text{but} \quad L_1 \neq L_2$$

x_0 is called a *jump discontinuity* of F.

If x_0 is neither removable nor jump, x_0 is called an *essential discontinuity* of F. In particular, if either $F(x_0^-)$ does not exist (but is not infinite) or $F(x_0^+)$ does not exist (but is not infinite), x_0 is called a *fundamental essential discontinuity* of F.

Remark. If x_0 is an endpoint of $[a,b]$. it is understood that the limits under discussion are the appropriate one-sided limits.

The key to our examination of a discontinuity of a derivative is Darboux's Theorem. For a proof, see [1].

Theorem 1.1. (Darboux) Suppose that f is differentiable on $[a,b]$ and that $f'(a) < f'(b)$. If $\gamma \in \Re$ is such that $f'(a) < \gamma < f'(b)$, then there exists $c \in (a,b)$ such that $f'(c) = \gamma$.

Remark. There is, of course, an analogous version for the case that $f'(a) > f'(b)$.

Remark. The Darboux property is frequently used to show that certain functions fail to possess antiderivatives. We refer the reader to the examples in section 3.

2. Discontinuous derivatives and fundamental essential discontinuities

Theorem 2.1. Let $f: [a,b] \to \Re$ be such that f is differentiable on $[a,b]$. Let $x_0 \in (a,b)$. If f' is discontinuous at x_0, then x_0 is a fundamental essential discontinuity of f'.

Proof. Our approach is to show that

(a) x_0 is not a removable discontinuity,
(b) x_0 is not a jump discontinuity,
(c) $f'(x_0^+) \neq \pm\infty$; $f'(x_0^-) \pm \infty$,

and then use (a)–(c) to conclude that x_0 is a fundamental essential discontinuity of f'.

So suppose first that x_0 is removable. Then there exists $L \in \Re$ such that $\lim_{x\to x_0} f'(x) = L \neq f'(x_0)$. Without loss of generality, $L < f'(x_0)$. Putting

$$\varepsilon = \frac{f'(x_0) - L}{2}$$

we have that there exists $\delta >; 0$ such that

$$|f'(x) - L| < \varepsilon \quad \text{for } x \in [a,b] \text{ for which } x_0 < x < x_0 + \delta$$

Noting that $L + \varepsilon < f'(x_0)$, let $\gamma \in (L + \varepsilon, f'(x_0))$ and choose $t \in (x_0, x_0 + \delta)$. Then f is differentiable on $[x_0, t]$ and since $f'(t) < L + \varepsilon$, $f'(t) < \gamma < f'(x_0)$. So by Theorem 1.1, there exists $c \in (x_0, t)$ such that $f'(c) = \gamma$. But $f'(c) < L + \varepsilon$ and since $\gamma > L + \varepsilon$, it follows that $f'(c) \neq \gamma$ and we have a contradiction. Therefore (a) holds.

Next, suppose that x_0 is a jump discontinuity of f'. Then $f'(x_0^-)$ and $f'(x_0^+)$ both exist in $\Re$ with $f'(x_0^-) \neq f'(x_0^+)$. Without loss of generality, assume that $f'(x_0^-) < f'(x_0^+)$. Put

$$\varepsilon = \frac{f'(x_0^+) - f'(x_0^-)}{3}$$

Note that $f'(x_0^-)+\varepsilon < f'(x_0^+)-\varepsilon$. Then there exists $\delta_1 > 0$ such that for $x \in [a,b]$ for which $x_0 - \delta_1 < x < x_0$,

$$|f'(x) - f'(x_0^-)| < \varepsilon \tag{1}$$

and there exists $\delta_2 > 0$ such that if $x \in [a,b]$ and $x_0 < x < x_0 + \delta_2$,

$$|f'(x) - f'(x_0^+)| < \varepsilon \tag{2}$$

Letting $\delta = \min\{\delta_1, \delta_2\}$, choose $\alpha, \beta \in [a,b]$ such that

$$x_0 - \delta < \alpha < x_0 < \beta < x_0 + \delta$$

and let $\gamma \neq f'(x_0)$ be such that

$$f'(\alpha) < f'(x_0^-) + \varepsilon < \gamma < f'(x_0^+) - \varepsilon < f'(\beta)$$

By Theorem 1.1, there exists $c \in (\alpha, \beta)$ such that $f'(c) = \gamma$. Since $\gamma \neq f'(x_0)$, $c \neq x_0$ and thus either $x_0 - \delta < c < x_0$ or $x_0 < c < x_0 + \delta$. In the former case, we have $x_0 - \delta_1 \leqslant x_0 - \delta < c < x_0$ and thus

$$|f'(c) - f'(x_0^-)| = \gamma - f'(x_0^-) > \varepsilon \tag{3}$$

In the latter, we have that $x_0 < c < x_0 + \delta \leqslant x_0 + \delta_2$ and so

$$|f'(c) - f'(x_0^+)| = f'(x_0^+) - \gamma > \varepsilon \tag{4}$$

Since inequality (3) contradicts (1) and inequality (4) contradicts (2), we conclude that (b) holds.

To show that (c) holds, we demonstrate that

$$f'(x_0^+) \neq +\infty$$

The demonstrations that $f(x_0^+) \neq -\infty$ and $f'(x_0^-) \neq \pm\infty$ are handled in a similar fashion. So suppose $f'(x_0^+) = +\infty$. Let $M > f'(x_0)$. Then there exists $\delta > 0$ such that for each $x \in [a,b]$ for which $x_0 < x < x_0 + \delta$, we have $f'(x) > M$. Choose $t \in (x_0, x_0 + \delta)$. Then f is differentiable on $[x_0, t]$ and $f'(x_0) < f'(t)$. Now let $\gamma \in \Re$ be such that $f'(x_0) < \gamma < M < f'(t)$. Then for each $c \in (x_0, t)$, $f'(c) > M > \gamma$ and we have a contradiction to Theorem 1.1. Hence (c) holds. By (a) and (b), $\lim_{x \to x_0} f'(x) \neq L$ for any $L \in \Re$ and $f'(x_0^+)$ and $f'(x_0^-)$ cannot both be finite. So at least one of $f'(x_0^+)$ or $f'(x_0^-)$ fails to exist. But by (c), neither is infinite. Thus x_0 is a fundamental essential discontinuity of f' and the proof of Theorem 2.1 is complete.

Remark. The word 'or' cannot be replaced by the word 'and' in the definition of fundamental essential discontinuity. For example, consider

$$f(x) = \begin{cases} x^2 \sin\dfrac{1}{x^2}, & x > 0 \\ 0, & x \leqslant 0 \end{cases}$$

which is differentiable on $\Re$ with

$$f'(x) = \begin{cases} 2x \sin\dfrac{1}{x^2} - \dfrac{2}{x}\cos\dfrac{1}{x^2}, & x > 0 \\ 0, & x \leqslant 0 \end{cases}$$

Since f is continuous at $x_0 = 0$ and $f'(0^-) = 0$,

$$f'_{-}(0) = \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} x\sin\frac{1}{x^2} = 0 = f'(0^-)$$

But

$$f'_{+}(0) = \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} x\sin\frac{1}{x^2} = 0$$

whereas

$$f'(0^+) = \lim_{x\to 0^+} 2x\sin\frac{1}{x^2} - \frac{2}{x}\cos\frac{1}{x^2}$$

which does not exist. In particular, for the positive sequences

$$x_n = \frac{1}{\sqrt{2n\pi}} \quad \text{and} \quad z_n = \frac{1}{\sqrt{(2n+1)\pi}}$$

respectively, we have that $\lim_{n\to\infty} = 0 = \lim_{n\to\infty} z_n$, while $\lim_{n\to\infty} f'(x_n) = \lim_{n\to\infty} -2\sqrt{2n\pi} = -\infty$ and $\lim_{n\to\infty} f'(z_n) = \lim_{n\to\infty} 2\sqrt{(2n+1)\pi} = +\infty$ (so that f' is unbounded on any right-neighbourhood of $x_0 = 0$). Although $x_0 = 0$ is indeed a fundamental essential discontinuity, we see that *at least one* of $f'(x_0^+)$ or $f'(x_0^-)$ *can* exist.

3. Examples

In this section, we offer examples pertaining to Theorems 1.1 and 2.1. In each example we seek to determine if the given function is the derivative of some real function.

Example 3.1. Let Q designate the set of rational numbers and let $f: [0,1] \to \Re$ be given by

$$f(x) = \begin{cases} 0, & x \in [0,1] \cap Q \\ 1, & x \in [0,1] \cap Q^c \end{cases}$$

Observe that for each $x_0 \in [0,1]$, neither $f(x_0^-)$ nor $f(x_0^+)$ exist. Thus, every $x \in [0,1]$ is a fundamental essential discontinuity of f. Now suppose that there exists $F: [0,1] \to \Re$ such that $F'(x) = f(x)$ for each $x \in [0,1]$. Let $\alpha = 0, \gamma = \frac{1}{2}$ and choose $\beta \in (\frac{1}{2}, 1) \cap Q^c$. Then $F'(\alpha) = f(\alpha) = 0 < \gamma < 1 = f(\beta) = F'(\beta)$ and so by Theorem 1.1, there exists $c \in (\alpha, \beta)$ such that $F'(c) = f(c) = \gamma = \frac{1}{2}$. But this is impossible since the range of f is $\{0,1\}$. We conclude that f has no antiderivative on $[0,1]$.

Example 3.2. Let $f: [-1,1] \to \Re$ be given by

$$f(x) = \begin{cases} 0, & x \in [-1,0) \\ 1, & x \in [0,1] \end{cases}$$

Since $f(0^-) = 0 \neq 1 = f(0^+)$, $x_0 = 0$ is a jump discontinuity of f. So by Theorem 2.1, f has no antiderivative on $[0,1]$.

Example 3.3. Let $f: [-1,1] \to \Re$ be given by

$$f(x) = \begin{cases} \sin(1/x), & x \in [-1,0] \cup (0,1] \\ 0, & x = 0 \end{cases}$$

and put $F(x) = \int_0^x f(t)\mathrm{d}t$, $x \in [-1,1]$. Since f is bounded and has a single discontinuity at $x_0 = 0$, f is Riemann integrable on $[-1,1]$; hence, F is well defined. Since f is continuous for $x \neq 0$, by the Second Fundamental Theorem of Calculus (see [2]), $F'(x) = f(x)$, $x \neq 0$. We claim also that $F'(0) = 0$. To see this, observe first that

$$F(x) = F(-x), \qquad x \in [-1,1] \tag{5}$$

Equation (5) is clear if $x = 0$. If $x \neq 0$ (putting $t = -u$), we have that

$$F(-x) = \int_0^{-x} \sin(1/t)\mathrm{d}t = -\int_0^x \sin(-1/u)\,\mathrm{d}u = \int_0^x \sin(1/u)\,\mathrm{d}u = F(x)$$

and again equation (5) holds.

Next, we show that

$$|F(x)| \leq 2x^2, \qquad x \in [-1,1], \tag{6}$$

Since $F(0) = 0$ and equation (5) holds, we need only show that inequality (6) holds for $x > 0$. For if so, and $x < 0$, then $-x > 0$ so that

$$|F(x)| = |F(-x)| \leqslant 2(-x)^2 = 2x^2$$

and again inequality (6) holds. To show that inequality (6) holds for $x > 0$, let $0 < y < x$. Then

$$\int_y^x \sin(1/t)\mathrm{d}t = \int_y^x t^2 \frac{\sin(1/t)}{t^2}\,\mathrm{d}t.$$

Applying Integration by Parts (put $u = t^2, v = \cos(1/t)$) we have that

$$\int_y^x t^2 \frac{\sin(1/t)}{t^2}\,\mathrm{d}t = [t^2 \cos(1/t)]_y^x - 2\int_y^x t\cos(1/t)\,\mathrm{d}t$$

$$= x^2\cos(1/x) - y^2\cos(1/y) - 2\int_y^x t\cos(1/t)\,\mathrm{d}t$$

Therefore,

$$\left|\int_y^x \sin(1/t)\,\mathrm{d}t\right| \leqslant x^2 + y^2 = 2\int_y^x t\,\mathrm{d}t$$

$$= x^2 + y^2 + x^2 - y^2 = 2x^2$$

Hence (see [2], Theorem 3.9)

$$\left|\int_0^x \sin(1/t)\,\mathrm{d}t\right| = \left|\lim_{y\to 0^+}\int_y^x \sin(1/t)\,\mathrm{d}t\right| = \lim_{y\to 0^+}\left|\int_y^x \sin(1/t)\,\mathrm{d}t\right| \leqslant 2x^2$$

and we conclude that inequality (6) holds for $x > 0$. Finally, for $x \in [-1,0) \cup (0,1]$, inequality (6) implies that

$$\left|\frac{F(x) - F(0)}{x - 0}\right| = \left|\frac{F(x)}{x}\right| \leqslant 2|x|$$

and it follows that $F'(0) = \lim_{x \to 0} (F(x)/x) = 0$. Note that neither $F'(0^+) = f(0^+)$ nor $F'(0^-) = f(0^-)$ exist and therefore $x_0 = 0$ is a fundamental essential discontinuity of $F' = f$.

Example 3.4. Suppose that $f: [a,b] \to \Re$ is continuous on $[a,b]$. Define $F: [a,b] \to \Re$ by

$$F(x) = \int_a^x f(t)\,\mathrm{d}t, \qquad x \in [a,b]$$

By the Second Fundamental Theorem of Calculus, F is differentiable on $[a,b]$ and $F'(x) = f(x)$ for each $x \in [a,b]$.

Example 3.5. Let $f: [-1,1] \to \Re$ be given by

$$f(x) = \begin{cases} x, & x \in [-1,0) \cup (0,1] \\ 1, & x = 0 \end{cases}$$

Note that $\lim_{x \to 0} f(x) = 0 \neq f(0)$. Since $x_0 = 0$ is a removable discontinuity of f, by Theorem 2.1, f fails to have an antiderivative on $[-1,1]$.

Example 3.6. Define $f: [-1,1] \to \Re$ by

$$f(x) = \begin{cases} 1/x, & x \in [-1,0) \cup (0,1] \\ 0, & x = 0 \end{cases}$$

Since $f(0^-) = -\infty$ and $f(0^+) = +\infty$, $x_0 = 0$ is not a fundamental essential discontinuity of f. Thus, by Theorem 2.1, f does not have an antiderivative on $[-1,1]$.

Remark. For an in-depth discussion of *how badly discontinuous* a derivative can be, see [3].

References

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Generating covariance matrices

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A simple procedure is highlighted for generating Covariance matrices.