RESPOSTAS DE EXERCÍCIOS DA LISTA 05.
PROF. MAURÍCIO ZAHN

01) Basta aplicar o T. do Sanduíche:

$$\left.\begin{array}{l} \bullet \displaystyle\lim_{x\to 0} 1-\cos^2 x = 1-\cos^2 0 = 1-1=0 \; ; \\ \bullet \displaystyle\lim_{x\to 0} x^2 = 0^2 = 0 \end{array}\right\} \Longrightarrow \lim_{x\to 0} f(x) = 0.$$

02) provar: $\displaystyle\lim_{x\to 0} x\cdot \operatorname{sen}\frac{1}{x} = 0.$

De fato, basta observar que

$$0 \leq \left| x\cdot\operatorname{sen}\frac{1}{x}\right| = |x|\cdot\underbrace{\left|\operatorname{sen}\frac{1}{x}\right|}_{\leq 1} \leq |x| \; ;$$

e como

$$\lim_{x\to 0} 0 = 0 = \lim_{x\to 0} |x| \; ;$$

pelo T. do Sanduíche segue que $\displaystyle\lim_{x\to 0} x\cdot\operatorname{sen}\frac{1}{x} = 0.$

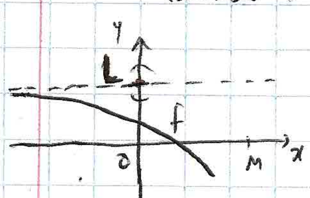
03) Basta usar o T. do Sanduíche.
Vamos concluir que

$$\lim_{x\to 0} h(x) = 1.$$

04) Mostrar que $\displaystyle\lim_{x\to -\infty} \frac{8x+3}{2x-1} = 4.$

Recordando: $\displaystyle\lim_{x\to -\infty} f(x) = L \overset{def}{\Longleftrightarrow} \forall \varepsilon>0, \exists M>0$ tal que,

$$\forall x < -M \Longrightarrow |f(x)-L| < \varepsilon.$$



Assim, dado $\varepsilon>0$, precisamos achar $M>0$ tal que, $\forall x<-M$, implique em

$$\left|\frac{8x+3}{2x-1} - 4\right| < \varepsilon.$$ Vamos analisar

$$\left|\frac{8x+3}{2x-1}-4\right| :$$

$$\left|\frac{8x+3}{2x-1}-4\right| = \left|\frac{8x+3-8x+4}{2x-1}\right| = \frac{7}{|2x-1|}.$$

Como $x \to -\infty$, então $|2x-1| = -(2x-1) = -2x+1$.

Sendo $x < -M \Rightarrow -2x > 2M \Rightarrow -2x+1 > 2M+1$, e

então $\dfrac{1}{-2x+1} < \dfrac{1}{2M+1}$, e daí:

$$\left|\frac{8x+3}{2x-1}-4\right| = \frac{7}{|2x-1|} = \frac{7}{-2x+1} < \underbrace{\frac{7}{2M+1}}_{:=\varepsilon}$$

Escrevendo $\varepsilon = \dfrac{7}{2M+1}$, isolando M, obtemos:

$$\varepsilon\cdot(2M+1) = 7 \Rightarrow 2\varepsilon M + \varepsilon = 7 \Rightarrow M = \frac{7-\varepsilon}{2\varepsilon} > 0$$

Portanto, encontramos $M>0$ tal que $|f(x)-4|<\varepsilon$, sempre que $x<-M$; o que prova o limite solicitado. $\square$

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05) Provaremos que $\lim\limits_{x\to+\infty} \tanh x = 1$. A outra é análoga.

Dado $\varepsilon > 0$, precisamos achar $M>0$ tal que, $\forall\, x > M \Rightarrow |f(x)-1|<\varepsilon$

Vamos avaliar $|f(x)-1|$:

$$|f(x)-1| = |\tanh x - 1| = \left|\frac{e^x-e^{-x}}{e^x+e^{-x}}-1\right| = \left|\frac{\cancel{e^x}-e^{-x}-\cancel{e^x}-e^{-x}}{e^x+e^{-x}}\right|$$

$$= \left|\frac{-2e^{-x}}{e^x+e^{-x}}\right| = \left|\frac{-\frac{2}{\cancel{e^x}}}{\frac{e^{2x}+1}{\cancel{e^x}}}\right| = \frac{2}{e^{2x}+1}.$$

Como vamos tomar $x > M$, observando que e^x é uma função crescente, segue que $2x > 2M \Rightarrow e^{2x} > e^{2M}$, e daí $e^{2x}+1 > e^{2M}+1$. Portanto, $\dfrac{1}{e^{2x}+1} < \dfrac{1}{e^{2M}+1}$; e daí:

$$|f(x) - 1| = \frac{2}{e^{2x}+1} < \underbrace{\frac{2}{e^{2M}+1}}_{:= \varepsilon}$$

Escrevendo $\varepsilon = \dfrac{2}{e^{2M}+1}$, vamos obter M:

$$\varepsilon . (e^{2M}+1) = 2 \Rightarrow \varepsilon . e^{2M} + \varepsilon = 2$$

$$\Rightarrow e^{2M} = \frac{2-\varepsilon}{\varepsilon}$$

$$\Rightarrow \ln e^{2M} = \ln\left(\frac{2-\varepsilon}{\varepsilon}\right)$$

$$\Rightarrow 2M = \ln\left(\frac{2-\varepsilon}{\varepsilon}\right) \Rightarrow \underline{M = \ln\sqrt{\frac{2-\varepsilon}{\varepsilon}}}.$$

Assim, dado $\varepsilon > 0$, $\exists\, M > 0$ (a saber, $M = \ln\sqrt{\dfrac{2-\varepsilon}{\varepsilon}}$) tal que, $\forall x > M \Rightarrow |\tanh x - 1| < \varepsilon$.

$\square$

O exercício terminou aqui. Apenas para ilustrar, vamos considerar o caso quando $\varepsilon = 0{,}1$. Então o $M > 0$ será $M = \ln\sqrt{\dfrac{2-0{,}1}{0{,}1}} \cong 1{,}47$.

Assim, $\forall x > 1{,}47$, é garantido que $|\tanh x - 1| < 0{,}1$.

Por exemplo, tomando $x = 1{,}5 > 1{,}47$:

$$|\tanh(1{,}5) - 1| = |0{,}905148 - 1| = 0{,}094852 < 0{,}1.$$

06) $f(x) = \left| \left| \dfrac{x^2}{1-x^2} \right| - 1 \right| - 1.$

Vamos fazer o gráfico de $g(x) = \dfrac{x^2}{1-x^2}$, e depois tomar $f = \big| |g| - 1 \big| - 1.$

zeros de $g(x)$: $g(x) = 0 \iff \dfrac{x^2}{1-x^2} = 0 \iff \boxed{x = 0}.$

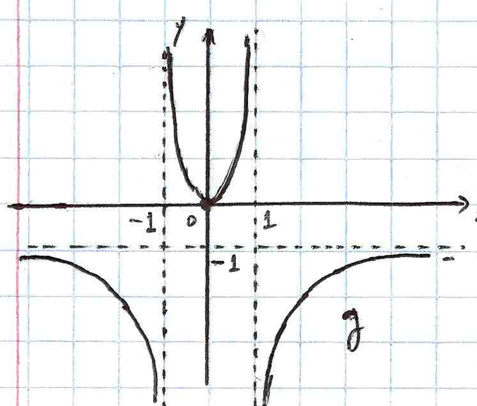
$D(g) = \mathbb{R} \setminus \{-1, 1\}$. (pois $1 - x^2 \neq 0$).

- $\lim\limits_{x \to -1^-} g(x) = \lim\limits_{x \to -1^-} \dfrac{x^2}{1-x^2} = \dfrac{1}{0^-} = -\infty.$
- $\lim\limits_{x \to -1^+} g(x) = \lim\limits_{x \to -1^+} \dfrac{x^2}{1-x^2} = \dfrac{1}{0^+} = +\infty.$
- $\lim\limits_{x \to 1^-} g(x) = \lim\limits_{x \to 1^-} \dfrac{x^2}{1-x^2} = \dfrac{1}{0^+} = +\infty.$
- $\lim\limits_{x \to 1^+} g(x) = \lim\limits_{x \to 1^+} \dfrac{x^2}{1-x^2} = \dfrac{1}{0^-} = -\infty.$

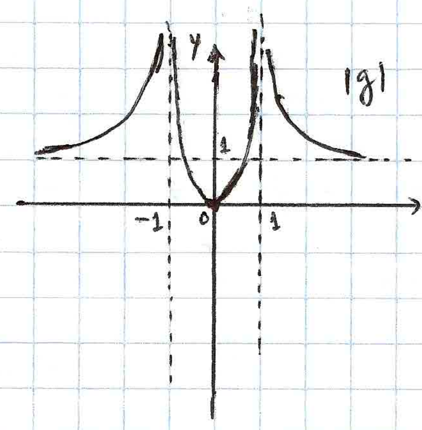
Assintotas verticais: $x = -1$ e $x = 1$.

- $\lim\limits_{x \to \pm\infty} g(x) = \lim\limits_{x \to \pm\infty} \dfrac{x^2}{1-x^2} = -1.$

Assintota horizontal: $y = -1$. Assim, temos:

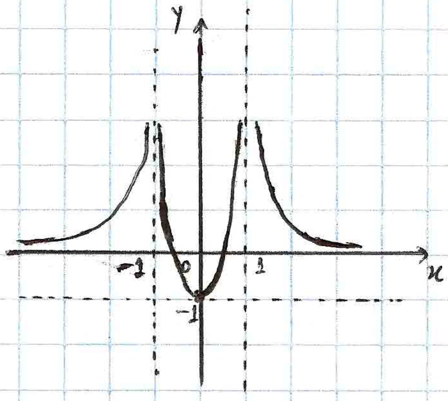


$|g|$: $\rightsquigarrow$

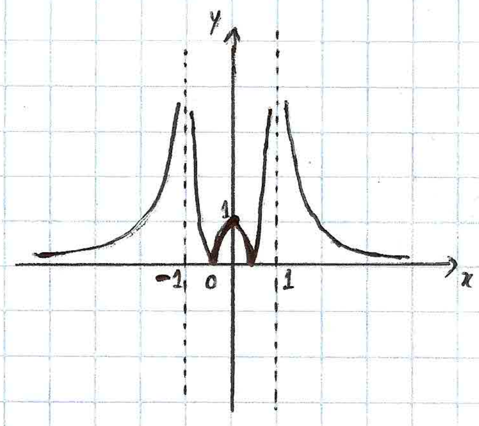


$|g| - 1$: $\rightsquigarrow$

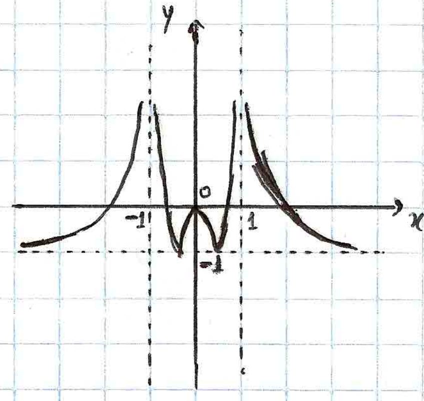
$|g|-1$:



$||g|-1|$:



$f = ||g|-1|-1$



$D(f) = \mathbb{R} \setminus \{-1, 1\}.$

$Im(f) = [-1, +\infty).$

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07) (f) $\lim\limits_{x \to a} \dfrac{\operatorname{sen} x - \operatorname{sen} a}{x-a} = \dfrac{0}{0}$ (INDETERM.)

$$\lim_{x \to a} \frac{\operatorname{sen} x - \operatorname{sen} a}{x-a} = \lim_{x \to a} \frac{2 \cdot \operatorname{sen}\left(\frac{x-a}{2}\right) \cdot \cos\left(\frac{x+a}{2}\right)}{x-a} =$$

$$\operatorname{sen} p - \operatorname{sen} q = 2 \cdot \operatorname{sen} \frac{p-q}{2} \cdot \cos \frac{p+q}{2}$$

$$= \lim_{x \to a} \frac{\cancel{2} \cdot \operatorname{sen}\left(\frac{x-a}{2}\right) \cdot \cos\left(\frac{x+a}{2}\right)}{\left(\frac{x-a}{2}\right) \cdot \cancel{2}} = \lim_{x \to a} \cos\left(\frac{x+a}{2}\right) = \cos a.$$

$\to 1$

(i) $\lim\limits_{x \to 0} (\sec x - \operatorname{sen} x)^{\frac{2}{x}} = 1^{\infty}$ (INDET.).

$$\lim_{x \to 0} (\sec x - \operatorname{sen} x)^{\frac{2}{x}} = \lim_{x \to 0} \left(1 + (\sec x - \operatorname{sen} x - 1)\right)^{\frac{2}{x}} =$$

$$= \Bigg[\underbrace{\lim_{x \to 0} \left(1 + (\sec x - \operatorname{sen} x - 1)\right)^{\frac{1}{\sec x - \operatorname{sen} x - 1}}}_{\to e}\Bigg]^{\frac{\sec x - \operatorname{sen} x - 1}{1} \cdot \frac{2}{x}} =$$

(segue)

$$= e^{\lim_{x\to 0} \frac{2.(\sec x - \text{sen} x - 1)}{x}} = e^{2.\left[\lim_{x\to 0} \frac{\sec x - 1}{x} - \lim_{x\to 0} \frac{\text{sen} x}{x}\right]} =$$

$$1 + \tan^2 x = \sec^2 x$$

$$\Rightarrow \tan^2 x = \sec^2 x - 1$$

$$\Rightarrow \tan^2 x = (\sec x + 1)(\sec x - 1)$$

$$\Rightarrow \sec x - 1 = \frac{\tan^2 x}{\sec x + 1}$$

$$= e^{2.\left[\lim_{x\to 0} \frac{\tan^2 x}{\sec x + 1} \cdot \frac{1}{x} - \lim_{x\to 0} \frac{\text{sen} x}{x}\right]} =$$

$$= e^{2.\left[\lim_{x\to 0} \underbrace{\frac{\tan x}{x}}_{1} \cdot \frac{\tan x}{\sec x + 1} - \lim_{x\to 0} \underbrace{\frac{\text{sen} x}{x}}_{1}\right]}$$

$$= e^{2.\left[\frac{0}{2} - 1\right]} = e^{-2}$$

(K) $\lim_{x\to 0} \frac{1}{ax^2+bx} . \ln(1 + c . \text{arcsen} x) = \infty . 0$ (INDETERM.)

Vamos transformar esta indeterm. em outra forma :

$$\lim_{x\to 0} \frac{1}{ax^2+bx} . \ln(1 + c.\text{arcsen} x) = \lim_{x\to 0} \ln(1 + c.\text{arcsen} x)^{\frac{1}{ax^2+bx}} =$$

$$= \ln\left(\lim_{x\to 0} (1 + c.\text{arcsen} x)^{\frac{1}{ax^2+bx}}\right) = \ln\left(\underbrace{1^{\infty}}_{\text{(INDETERM.)}}\right)$$

$$= \ln\left[\underbrace{\lim_{x\to 0} (1 + c\,\text{arcsen} x)^{\frac{1}{c\,\text{arcsen} x}}}_{e}\right]^{\frac{c.\text{arcsen} x}{1} \cdot \frac{1}{ax^2+bx}} =$$

$$= \ln e^{\lim_{x\to 0} \frac{c.\text{arcsen} x}{\underbrace{x.}_{1}(ax+b)}} = \ln e^{\frac{c}{b}} = \frac{c}{b} . \ln e = \frac{c}{b} .$$

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08) (a) $f(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & \text{se } x<2 \\ x^2-3x+6, & \text{se } x \geq 2 \end{cases}$ em $x=2$.

(i) $\exists\, f(2) = (2)^2 - 3\cdot(2) + 6 = 4$

(ii) $\exists \lim\limits_{x\to 2} f(x)$? Para ver se existe, vamos estudar os limites laterais.

- $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} \dfrac{x^2-4}{x-2} = \lim\limits_{x\to 2^-} \dfrac{(x+2)\cancel{(x-2)}}{\cancel{x-2}} = 4$

- $\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} x^2-3x+6 = 4-6+6 = 4$

Como $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^+} f(x) = 4$, segue que $\exists \lim\limits_{x\to 2} f(x) = 4$.

(iii) $\lim\limits_{x\to 2} f(x) = f(2) = 4$.

Logo, de (i), (ii) e (iii) concluímos que f é contínua em $x=2$.

(b) $f(x) = \begin{cases} \dfrac{1-\cos 2x}{x^2}, & \text{se } x<0 \\ x+2, & \text{se } x \geq 0 \end{cases}$ em $x=0$:

(i) $\exists\, f(0) = 0+2 = 2$

(ii) $\exists \lim\limits_{x\to 0} f(x)$? Vamos investigar os limites laterais:

- $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} \dfrac{1-\cos 2x}{x^2} = \dfrac{0}{0}$. (INDET.)

Como $1-\cos 2x = \dfrac{\operatorname{sen}^2 2x}{1+\cos 2x}$, segue que:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0} \frac{1-\cos 2x}{x^2} = \lim_{x\to 0^-} \frac{\operatorname{sen}^2 2x}{1+\cos 2x}\cdot\frac{1}{x^2} =$$

$$= \lim_{x\to 0^-} \underbrace{\frac{\operatorname{sen} 2x}{2x}}_{1}\cdot\underbrace{\frac{\operatorname{sen} 2x}{2x}}_{1}\cdot\frac{4}{1+\cos 2x} = \frac{4}{2} = 2.$$

• $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} x+2 = 0+2 = 2$

Como $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} f(x) = 2$, segue que $\exists \lim\limits_{x\to 0} f(x) = 2$.

(iii) $\lim\limits_{x\to 0} f(x) = 2 = f(0)$.

Logo, por (i), (ii) e (iii) segue que f é contínua no ponto $x = 0$.

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10) Como $f(x) = \begin{cases} \dfrac{x^2-9}{x-3} & , \text{se } x < 3 \\ mx+1 & , \text{se } x \geq 3 \end{cases}$ já é contínua em $x=3$ (por hipótese); segue que

$$\exists \lim_{x\to 3} f(x) = f(3) \; ;$$

onde $f(3) = 3m+1$, e como $\exists \lim\limits_{x\to 3} f(x)$, segue que

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) \quad , \text{ou seja,}$$

$$\lim_{x\to 3^-} \frac{x^2-9}{x-3} = \lim_{x\to 3^+} mx+1$$

$$\Rightarrow \lim_{x\to 3^-} \frac{(x+3)\cancel{(x-3)}}{\cancel{x-3}} = \lim_{x\to 3^+} mx+1$$

$$\Rightarrow 3+3 = 3m+1 \Rightarrow 3m+1 = 6 \Rightarrow \boxed{m = \frac{5}{3}}$$
