LISTA 04 — RESPOSTAS DE ALGUNS EXERCÍCIOS
PROF. MAURÍCIO ZAHN.

01) (d) $\lim_{x \to +\infty} \frac{\sqrt[3]{x^2+1}}{x+1}$. Dividindo numerador e denominador por x,

e lembrando que $x = \sqrt[3]{x^3}$, temos:

$$\lim_{x \to +\infty} \frac{\sqrt[3]{x^2+1}}{x+1} = \lim_{x \to +\infty} \frac{\frac{1}{x}\sqrt[3]{x^2+1}}{\frac{1}{x}\cdot(x+1)} = \lim_{x \to +\infty} \frac{\sqrt[3]{\frac{x^2}{x^3}+\frac{1}{x^3}}}{\frac{x}{x}+\frac{1}{x}} =$$

$$= \lim_{x \to +\infty} \frac{\sqrt[3]{\frac{1}{x}+\frac{1}{x^3}}}{1+\frac{1}{x}} = \frac{0}{1} = 0.$$

(h) $\lim_{x \to 3} \frac{1}{x^2-9} = \frac{1}{0}$. Neste caso, precisamos estudar os

limites laterais:

- $\lim_{x \to 3^+} \frac{1}{x^2-9} = \lim_{x \to 3^+} \frac{1}{(x+3)(x-3)} = \frac{1}{6.(0^+)} = +\infty$
- $\lim_{x \to 3^-} \frac{1}{x^2-9} = \lim_{x \to 3^-} \frac{1}{(x+3).(x-3)} = \frac{1}{6.(0^-)} = -\infty$

Como $\lim_{x \to 3^-} f(x) \neq \lim_{x \to 3^+} f(x)$, concluímos que $\nexists \lim_{x \to 3} \frac{1}{x^2-9}$.

(o) $\lim_{x \to 0} \frac{1-\sqrt{\cos x}}{x^2} = \frac{0}{0}$ (INDETERMINAÇÃO).

Racionalizando, vamos obter:

$$\lim_{x \to 0} \frac{1-\sqrt{\cos x}}{x^2} = \lim_{x \to 0} \frac{1-\sqrt{\cos x}}{x^2} \cdot \frac{1+\sqrt{\cos x}}{1+\sqrt{\cos x}} = \lim_{x \to 0} \frac{1-\cos x}{x^2\cdot(1+\sqrt{\cos x})} \circledast$$

Como $\operatorname{sen}^2 x + \cos^2 x = 1$; temos $\operatorname{sen}^2 x = 1 - \cos^2 x$

$$\Rightarrow \operatorname{sen}^2 x = (1-\cos x)\cdot(1+\cos x)$$

$$\Rightarrow 1-\cos x = \frac{\operatorname{sen}^2 x}{1+\cos x}$$; e assim, temos que $\circledast$

fica:

(...)

$$\overset{=}{\ } \lim_{x\to 0} \frac{\frac{\operatorname{sen}^2 x}{1+\cos x}}{x^2(1+\sqrt{\cos x})} = \lim_{x\to 0} \underbrace{\left(\frac{\operatorname{sen} x}{x}\right)^2}_{\to 1} \cdot \frac{1}{(1+\cos x)(1+\sqrt{\cos x})} =$$

$$= \lim_{x\to 0} \frac{1}{(1+\cos x)(1+\sqrt{\cos x})} = \frac{1}{(1+1)(1+\sqrt{1})} = \frac{1}{4}$$

(u) $\displaystyle \lim_{x\to -\infty} \left(\frac{x^2-3x+2}{x^2-2x+4}\right)^{\frac{1-2x^2}{3x-1}}$

Note que a base tende para "1" e o expoente tende para "$+\infty$", quando $x \to -\infty$, ou seja, obtemos a indeterminação 1^∞. Assim:

$$\lim_{x\to -\infty} \left(\frac{x^2-3x+2}{x^2-2x+4}\right)^{\frac{1-2x^2}{3x-1}} = \lim_{x\to -\infty} \left(1 + \frac{x^2-3x+2}{x^2-2x+4} - 1\right)^{\frac{1-2x^2}{3x-1}} =$$

$$= \lim_{x\to -\infty} \left(1 + \frac{\cancel{x^2}-3x+2\cancel{-x^2}+2x-4}{x^2-2x+4}\right)^{\frac{1-2x^2}{3x-1}} = \lim_{x\to -\infty} \left(1 + \frac{-x-2}{x^2-2x+4}\right)^{\frac{1-2x}{3x-1}} =$$

$$= \Bigg[\underbrace{\lim_{x\to -\infty} \left(1 + \frac{-x-2}{x^2-2x+4}\right)^{\frac{x^2-2x+4}{-x-2}}}_{\to e}\Bigg]^{\frac{(-x-2)}{x^2-2x+4} \cdot \frac{1-2x^2}{3x-1}} = e^{\lim_{x\to -\infty} \frac{-x+2x^3-2+4x^2}{3x^3-6x^2+12x-x^2+2x-4}} =$$

$$= e^{\lim_{x\to -\infty} \frac{2\cancel{x^3}}{3\cancel{x^3}}} = e^{\frac{2}{3}}.$$

---

(4) $\lim\limits_{x\to 0} \frac{1}{x^2} \cdot \ln \sec x = \frac{0}{0}$ (INDETERM)

$$\lim_{x\to 0} \frac{1}{x^2} \cdot \ln \sec x = \lim_{x\to 0} \ln (\sec x)^{\frac{1}{x^2}} = \ln \underbrace{\lim_{x\to 0} (\sec x)^{\frac{1}{x^2}}}_{\text{“}1^{\infty}\text{”}} =$$

$$= \ln \lim_{x\to 0} (1 + \sec x - 1)^{\frac{1}{x^2}} = \ln \lim_{x\to 0} \underbrace{(1 + \sec x - 1)^{\frac{1}{\sec x - 1}}}_{\to e}{}^{\cdot \frac{\sec x - 1}{x^2}} =$$

$$= \ln e^{\lim\limits_{x\to 0} \frac{\sec x - 1}{x^2}} = \lim_{x\to 0} \frac{\sec x - 1}{x^2} \overset{(*)}{=} \frac{0}{0} \text{ (INDETERM.)};$$

e daí, como $1 + \tan^2 x = \sec^2 x \Rightarrow \sec^2 x - 1 = \tan^2 x$

$$\Rightarrow (\sec x + 1).(\sec x - 1) = \tan^2 x$$

$$\Rightarrow \sec x - 1 = \frac{\tan^2 x}{\sec x + 1} \text{ ; e daí:}$$

$$\overset{(*)}{=} \lim_{x\to 0} \frac{\sec x - 1}{x^2} = \lim_{x\to 0} \frac{\frac{\tan^2 x}{\sec x + 1}}{x^2} = \lim_{x\to 0} \frac{\tan^2 x}{x^2} \cdot \frac{1}{(\sec x + 1)} =$$

$$= \lim_{x\to 0} \underbrace{\left(\frac{\tan x}{x}\right)^2}_{1} \cdot \frac{1}{(\sec x + 1)} = (1)^2 \cdot \frac{1}{(1+1)} = \frac{1}{2} //$$

---

02) $a>0$; $a \neq 1$. Mostra que $\lim\limits_{x \to 0} \dfrac{a^x - 1}{x} = \ln a$.

<u>Solução:</u> Escreva $y = a^x - 1 \Rightarrow 1 + y = a^x$, e daí

$$x = \frac{\ln(1+y)}{\ln a};$$

e como $x \to 0 \Rightarrow y = a^x - 1 \to 1 - 1 = 0$, fazendo a mudança de variável, vamos obter:

$$\lim_{x \to 0} \frac{a^x - 1}{x} = \lim_{y \to 0} \frac{y}{\frac{\ln(1+y)}{\ln a}} = \lim_{y \to 0} \frac{y}{\ln(1+y)} \cdot \ln a =$$

$$= \lim_{y \to 0} \frac{1}{\frac{1}{y} \cdot \ln(1+y)} \cdot \ln a = \ln a \cdot \lim_{y \to 0} \frac{1}{\ln(1+y)^{\frac{1}{y}}} =$$

$$= \ln a \cdot \frac{1}{\lim\limits_{y \to 0} \ln(1+y)^{\frac{1}{y}}} = \ln a \cdot \frac{1}{\ln \underbrace{\lim\limits_{y \to 0} (1+y)^{\frac{1}{y}}}_{e}} =$$

$$= \ln a \cdot \frac{1}{\underbrace{\ln e}_{=1}} = \ln a. \quad //$$

$\square$.

---

03)

(c) $\lim\limits_{x \to 0} \dfrac{\operatorname{senh} x}{x} = \dfrac{0}{0}$ (INDET.)

$$\lim_{x \to 0} \frac{\operatorname{senh} x}{x} = \lim_{x \to 0} \frac{\frac{e^x - e^{-x}}{2}}{x} = \lim_{x \to 0} \frac{e^x - e^{-x}}{2x} =$$

$$= \lim_{x \to 0} \frac{e^x - 1 + 1 - e^{-x}}{2x} = \lim_{x \to 0} \frac{e^x - 1}{2x} + \lim_{x \to 0} \frac{-e^{-x} + 1}{2x} =$$

$$= \frac{1}{2} \lim_{x \to 0} \frac{e^x - 1}{x} + \frac{1}{2} \lim_{x \to 0} \frac{e^{-x} - 1}{-x} = \frac{1}{2} \ln e + \frac{1}{2} \ln e = 1. \quad //$$

04) (e) $f(x) = \dfrac{x^2-4}{1-x^2}$.

$D(f) = \mathbb{R} \setminus \{-1, 1\}$.

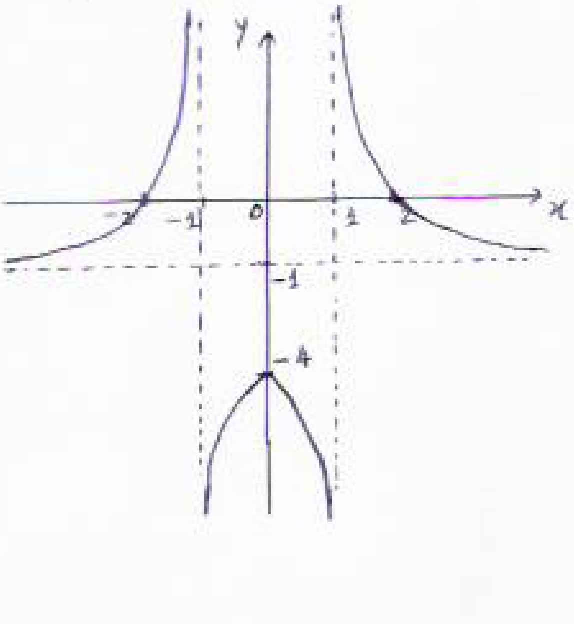
zeros de f: $f(x)=0 \iff \dfrac{x^2-4}{1-2x} = 0 \iff \boxed{x = \pm 2}$

- $\displaystyle\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} \frac{x^2-4}{(1-x)(1+x)} = \frac{-3}{(+2)\cdot(0^-)} = +\infty$
- $\displaystyle\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} \frac{x^2-4}{(1-x)(1+x)} = \frac{-3}{(2)\cdot(0^+)} = -\infty$
- $\displaystyle\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{x^2-4}{(1-x)(1+x)} = \frac{-3}{0^+\cdot(2)} = -\infty$
- $\displaystyle\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{x^2-4}{(1-x)(1+x)} = \frac{-3}{(0^-)\cdot 2} = +\infty$
- $\displaystyle\lim_{x\to \pm\infty} f(x) = \lim_{x\to \pm\infty} \frac{x^2-4}{1-x^2} = \lim_{x\to \pm\infty} \frac{x^2}{-x^2} = \lim_{x\to \pm\infty} -1 = -1$

ASSÍNTOTA HORIZONTAL: $y = -1$.

ASSÍNTOTAS VERTICAIS: $x = -1$ e $x = 1$.

Portanto, com as informações acima, temos que o esboço gráfico de f será: (note que também calculamos $f(0) = -4$ para termos uma informação a mais em $(-1,1)$ para o traçado gráfico de f).



$Im(f) = (-\infty, -4] \cup (-1, +\infty)$.

04) (f) Primeiramente faremos o gráfico de $g(x) = \dfrac{2x-5}{x^2-1}$, e depois "passaremos o módulo" para determinar $f = |g|$.

$$g(x) = \frac{2x-5}{x^2-1}.$$

$$x^2-1 \neq 0 \iff x \neq \pm 1$$

Logo, $D(g) = \mathbb{R} \setminus \{-1, 1\}$.

Assim, em $x=-1$ e $x=1$ temos duas assíntotas verticais.

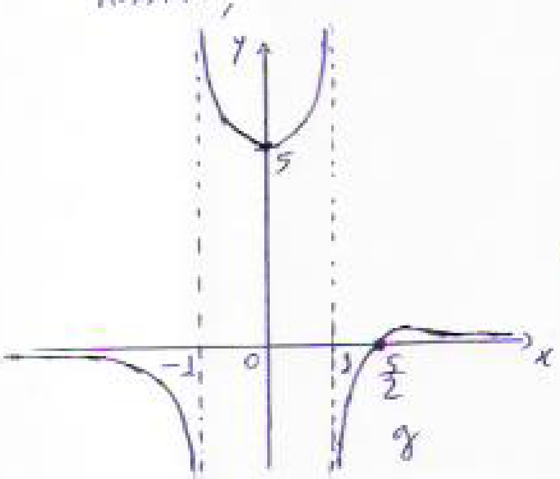
zeros de g: $g(x) = 0 \iff \dfrac{2x-5}{x^2-1} = 0 \iff$

$\iff 2x-5 = 0 \iff \boxed{x = \frac{5}{2}}$

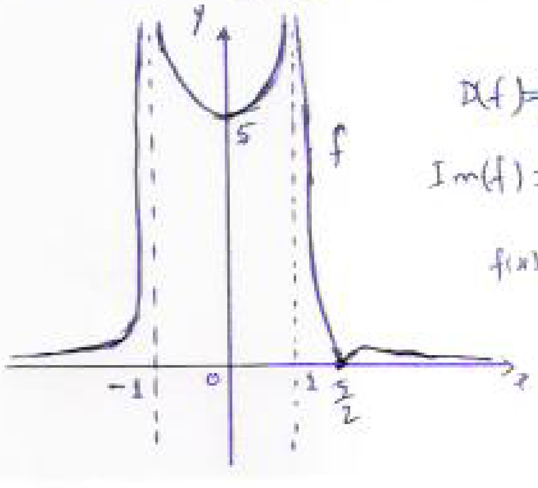
- $\lim\limits_{x \to -1^-} g(x) = \lim\limits_{x \to -1^-} \dfrac{2x-5}{x^2-1} = \lim\limits_{x \to -1^-} \dfrac{2x-5}{(x+1)\cdot(x-1)} = \dfrac{-7}{(0^-)\cdot(-2)} = -\infty.$
- $\lim\limits_{x \to -1^+} g(x) = \lim\limits_{x \to -1^+} \dfrac{2x-5}{(x+1)\cdot(x-1)} = \dfrac{-7}{(0^+)\cdot(-2)} = +\infty.$
- $\lim\limits_{x \to 1^-} g(x) = \lim\limits_{x \to 1^-} \dfrac{2x-5}{(x+1)\cdot(x-1)} = \dfrac{-3}{2\cdot(0^-)} = +\infty.$
- $\lim\limits_{x \to 1^+} g(x) = \lim\limits_{x \to 1^+} \dfrac{2x-5}{(x+1)\cdot(x-1)} = \dfrac{-3}{2\cdot(0^+)} = -\infty.$
- $\lim\limits_{x \to +\infty} g(x) = \lim\limits_{x \to +\infty} \dfrac{2x-5}{x^2-1} = \lim\limits_{x \to +\infty} \dfrac{2\cancel{x}}{x^{\cancel{2}}} = \lim\limits_{x \to +\infty} \dfrac{2}{x} = 0^+.$
- $\lim\limits_{x \to -\infty} g(x) = \lim\limits_{x \to -\infty} \dfrac{2x-5}{x^2-1} = \lim\limits_{x \to -\infty} \dfrac{2}{x} = 0^-.$

ASSÍNTOTA HORIZONTAL: $y = 0$.

Assim, e observando que $g(0) = 5$, obtemos o gráfico de g:



PASSANDO O MÓDULO OBTEMOS O GRÁFICO DE f:



$D(f) = \mathbb{R} \setminus \{-1, 1\}$

$Im(f) = [0, +\infty).$

$f(x) = |g(x)|$