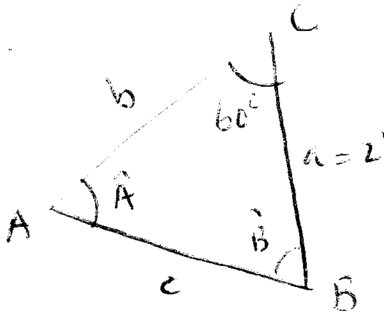
TRIGONOMETRIA. PROF. MAURÍCIO ZAHN

RESOLUÇÃO DE EXERCÍCIOS LISTAS 06 A 09

Lista 06.

07)




Achar Â e B̂.

Pela lei dos senos:

$$\frac{2b}{\sin\hat{A}} = \frac{b}{\sin\hat{B}} = \frac{c}{\sin 60°} \quad (*)$$

Pela lei dos cossenos:

$$c^2 = b^2 + (2b)^2 - 2\cdot b\cdot(2b)\cos 60°$$

$$c^2 = b^2 + 4b^2 - 4b^2\,\frac{1}{2} \Rightarrow c^2 = 3b^2 \Rightarrow \boxed{c = b\sqrt{3}}$$

Assim, voltando em (*), obtemos:

$$\frac{b}{\sin\hat{B}} = \frac{b\sqrt{3}}{\sin 60°} \Rightarrow \frac{1}{\sin\hat{B}} = \frac{\sqrt{3}}{\frac{\sqrt{3}}{2}} = \sqrt{3}\cdot\frac{2}{\sqrt{3}} = 2$$

$$\Rightarrow \sin\hat{B} = \frac{1}{2} \Rightarrow \begin{cases} \hat{B} = 30° \text{ ou} \\ \hat{B} = 150° \end{cases}$$

(mas $\hat{B} = 150°$ não serve pois $\hat{A} + 60° + \hat{B} = 180°$ [illegible] 180)

Portanto, $\hat{B} = 30°$ e disso,

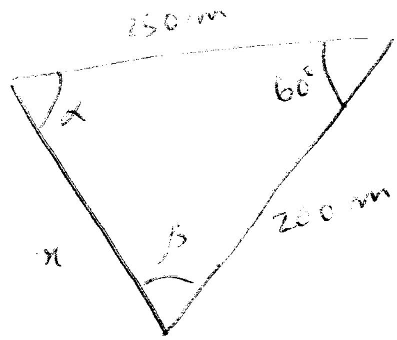
$$30° + 60° + \hat{A} = 180° \Rightarrow \boxed{\hat{A} = 90°}$$

Obs: se quiser, pode calcular Â da seguinte forma:

$$\frac{2b}{\sin\hat{A}} = \frac{b\sqrt{3}}{\sin 60°} = \frac{\sqrt{3}}{\frac{\sqrt{3}}{2}} = 2 \Rightarrow \frac{1}{\sin\hat{A}} = 1 \Rightarrow$$

$$\sin\hat{A} = 1 \Rightarrow \boxed{\hat{A} = 90°}$$

c)




Achar $\alpha$ e $\beta$.

Pela lei dos cossenos:

$$x^2 = (250)^2 + (200)^2 - 2 \cdot 250 \cdot 200 \cdot \cos 60°$$

$$x^2 = 62500 + 40000 - 50000$$

$$x^2 = 52500$$

$$\Longrightarrow \quad x = \sqrt{52500} = \sqrt{10^2 \cdot 5^2 \cdot 21} = 50\sqrt{21} \text{ m}$$

Assim, pela lei dos senos, temos:

$$\frac{50\sqrt{21}}{\operatorname{sen} 60°} = \frac{250}{\operatorname{sen}\beta} = \frac{200}{\operatorname{sen}\alpha}$$

•) $$\frac{50\sqrt{21}}{\frac{\sqrt{3}}{2}} = \frac{250}{\operatorname{sen}\beta} \iff 50\cancel{\sqrt{3}}\cdot\sqrt{7}\cdot\frac{2}{\cancel{\sqrt{3}}} = \frac{250}{\operatorname{sen}\beta}$$

$$\iff 2\sqrt{7} = \frac{5}{\operatorname{sen}\beta}$$

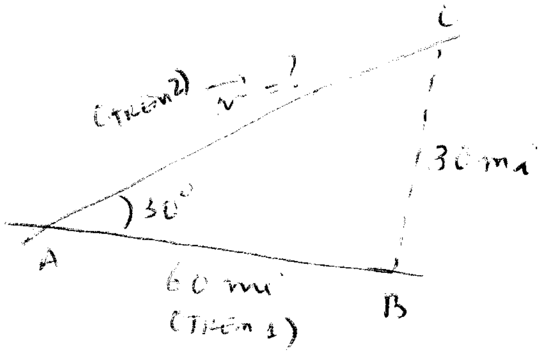
$$\iff \operatorname{sen}\beta = \frac{5}{2\sqrt{7}} \iff \beta = \arcsin\left(\frac{5}{2\sqrt{7}}\right)$$

•) $$\frac{50\sqrt{21}}{\operatorname{sen} 60°} = \frac{200}{\operatorname{sen}\alpha} \iff 50\cancel{\sqrt{3}}\sqrt{7}\cdot\frac{2}{\cancel{\sqrt{3}}} = \frac{200}{\operatorname{sen}\alpha}$$

$$\iff \frac{\sqrt{7}}{2} = \frac{1}{\operatorname{sen}\alpha} \iff \operatorname{sen}\alpha = \frac{2}{\sqrt{7}}$$

$$\iff \alpha = \arcsin\left(\frac{2}{\sqrt{7}}\right)$$

1.2)




$t = 3h$. a de velocidade $20\, mi/h$ andou $60\, mi$

Seja $x = \overline{AC}$. Assim, pela lei dos cossenos

$$30^2 = x^2 + (60)^2 - 2 \cdot x \cdot 60 \cdot \cos 30^\circ$$

$$900 = x^2 + 3600 - 120\, x \cdot \frac{\sqrt{3}}{2}$$

$$\Longleftrightarrow \quad x^2 - 60\sqrt{3}\, x - 900 = 0$$

$$x = \frac{60\sqrt{3} \pm \sqrt{10.800 + 3600}}{2 \cdot 1}$$

$$x = \frac{60\sqrt{3} \pm 120}{2} = 30\sqrt{3} \pm 60$$

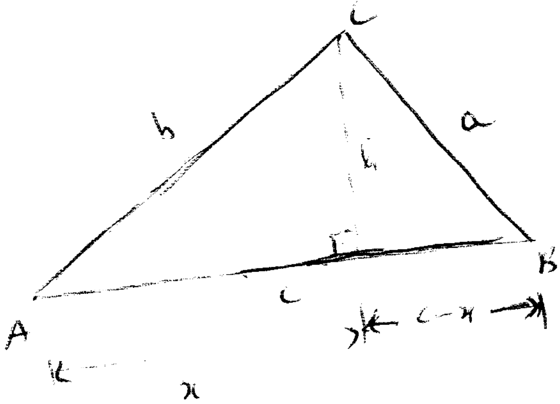
Como $x > 0$ ; $x = (30\sqrt{3} + 60)\, mi$

Ou seja, este trem (TREM 2) percorreu $(60 + 30\sqrt{3})\, mi$ em 3 horas, i.e., sua velocidade média $\bar{v}$ foi de:

$$\bar{v} = \frac{60 + 30\sqrt{3}}{3} = (20 + 10\sqrt{3})\, mi/h.$$

13) Vamos tomar as alturas relativas a cada lado e efetuar as seguintes decomposições (calculando simplesmente o cosseno de triângulos retângulos)

1º.



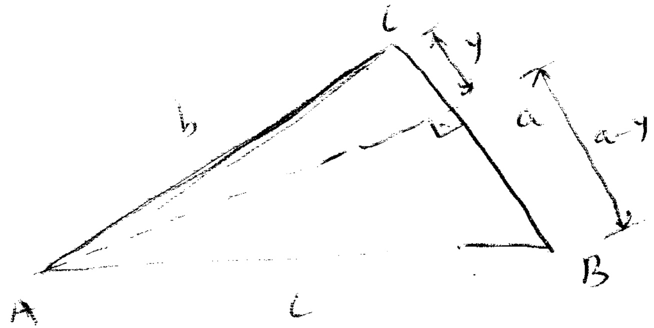

$$\cos A = \frac{x}{b} \Rightarrow x = b\cdot\cos A$$

$$\cos B = \frac{c-x}{a} \Rightarrow c - x = a\cos B$$

$$\Rightarrow x = c - a\cos B$$

Logo: $\boxed{b\cdot\cos A = c - a\cos B}$ (I)

2º.



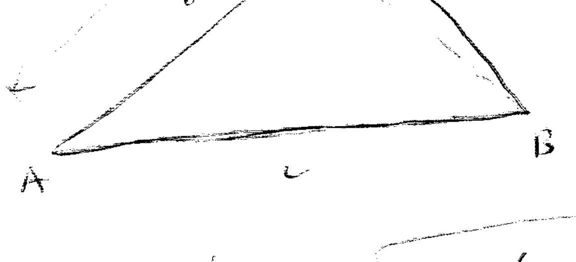

$$\cos C = \frac{y}{b} \Rightarrow y = b\cdot\cos C$$

$$\cos B = \frac{a-y}{c} \Rightarrow c\cos B = a - y$$

$$\Rightarrow y = a - c\cos B$$

Logo: $\boxed{b\cdot\cos C = a - c\cos B}$ (II)

3º.




$$\cos C = \frac{l}{a} \Rightarrow l = a\cdot\cos C$$

$$\cos A = \frac{b-l}{c} \Rightarrow b - l = c\cos A$$

$$\Rightarrow l = b - c\cos A$$

Logo: $\boxed{a\cdot\cos C = b - c\cos A}$ (III)

Juntando (I), (II) e (III) obtemos:

$$
\begin{array}{rl}
 & b\cdot\cos A = c - a\cdot\cos B \\
 & b\cdot\cos C = a - c\cdot\cos B \\
+ & a\cdot\cos C = b - c\cdot\cos A \\
\hline
\end{array}
$$

$$b\cdot\cos A + b\cdot\cos C + a\cdot\cos C = (a+b+c) - a\cdot\cos B - c\cdot\cos B - c\cdot\cos A$$

Isolando $(a+b+c)$ adequadamente, obteremos a igualdade desejada. $\square$

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<u>LISTA 07:</u>

02) (g) $f(x) = 1 - 2\cdot\csc\left(\frac{2\pi}{3} - 2x\right)$

<u>1º:</u> Consideremos o gráfico de $y = 1 - 2\cdot\text{sen}\left(\frac{2\pi}{3} - 2x\right)$

Façamos $\frac{2\pi}{3} - 2x = t \Rightarrow \frac{2\pi}{3} - t = 2x \Rightarrow \boxed{x = \frac{2\pi}{6} - \frac{t}{2}}$

$y = 1 - 2\cdot\text{sen}\, t$

| $t$ | $y = 1 - 2\,\text{sen}\,t$ |
|---|---|
| $0$ | $1$ |
| $\frac{\pi}{2}$ | $-1$ |
| $\pi$ | $1$ |
| $\frac{3\pi}{2}$ | $3$ |
| $2\pi$ | $1$ |

$x = \frac{2\pi}{6} - \frac{t}{2}$

$\frac{2\pi}{6} = \frac{4\pi}{12}$

$\frac{2\pi}{6} - \frac{\pi}{4} = \frac{4\pi - 3\pi}{12} = \frac{\pi}{12}$

$\frac{2\pi}{6} - \frac{\pi}{2}\times\frac{3}{3} = \frac{-2\pi}{12}$

$\frac{2\pi}{6} - \frac{3\pi}{4} = \frac{4\pi - 9\pi}{12} = -\frac{5\pi}{12}$

$\frac{2\pi}{6} - \pi\times\frac{6}{6} = \frac{-4\pi}{6} = \frac{-8\pi}{12}$

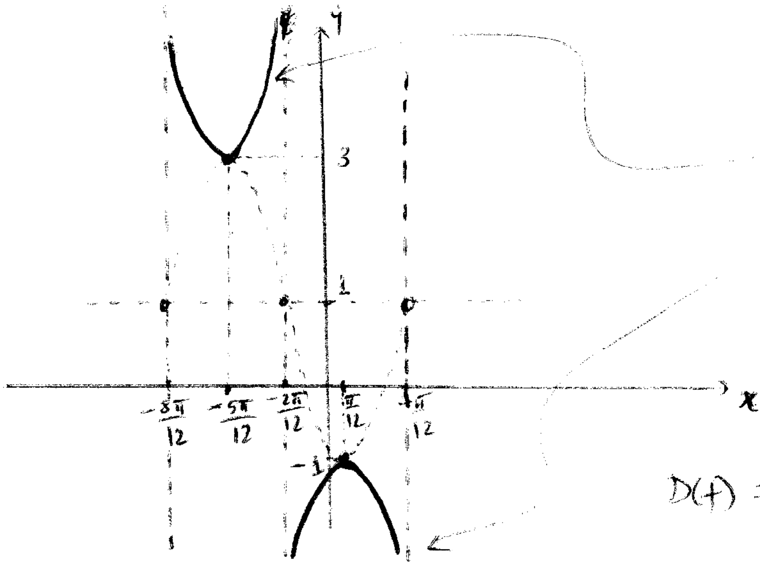


gráfico de

$y = 1 - 2 \cdot \csc\left(\frac{2\pi}{3} \ldots\right.$

$D(f) = \{ \mathbb{R} \setminus x \ldots$

$D(f) = \{x \in \mathbb{R} : \frac{2\pi}{3} - 2x \neq k\pi, \ldots$

$$Im(f) = (-\infty, -1] \cup [3, +\infty)$$

$$p = \frac{4\pi}{12} - \left(-\frac{8\pi}{12}\right) = \pi.$$

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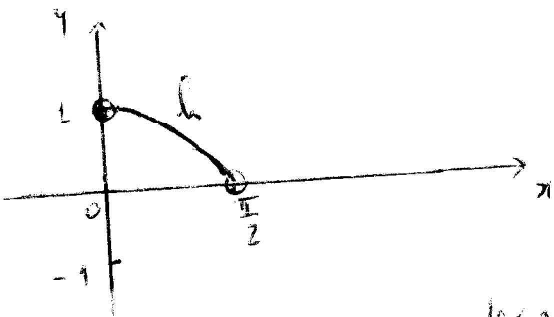
05) $h = f \circ g : (0, \frac{\pi}{2}) \longrightarrow (0, +\infty)$ é dada por:

$$h(x) = (f \circ g)(x) = f(g(x)) = f(2 \cdot \ln \csc x) =$$

$$= f(\ln \csc^2 x) = \sqrt{1 - \frac{1}{e^{\ln \csc^2 x}}}$$

$$= \sqrt{1 - \frac{1}{\csc^2 x}} = \sqrt{1 - \operatorname{sen}^2 x} = \cos x.$$




h não é periódica pois $D(h) \neq \mathbb{R}$.

h é injetiva pois é decrescente, mas não é sobrejetiva, pois $Im(h) = (0,1) \neq (0, \ldots$

Logo, h não é bijetiva

LISTA 08:

02) (c) $\tan\left(\operatorname{arcsen}\left(-\frac{2}{3}\right)+\operatorname{arcsen}\frac{1}{4}\right)$

Escreva $\alpha = \operatorname{arcsen}\left(-\frac{2}{3}\right)$ e $\beta = \operatorname{arcsen}\frac{1}{4}$.

Então $\begin{cases} \operatorname{sen}\alpha = -\frac{2}{3}, & \alpha \in 4^{o}q \text{ e} \\ \operatorname{sen}\beta = \frac{1}{4}, & \beta \in 1^{o}q. \end{cases}$

Queremos determinar $\tan(\alpha+\beta)$, onde:

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \cdot \tan\beta}$$

Como $\operatorname{sen}\alpha = -\frac{2}{3}$ e $\alpha \in 4^{o}q$, obtemos

$$\cos\alpha = -\sqrt{1-\frac{4}{9}} = -\frac{\sqrt{5}}{3},$$

e como $\operatorname{sen}\beta = \frac{1}{4}$, $\beta \in 1^{o}q$, obtemos

$$\cos\beta = +\sqrt{1-\frac{1}{16}} = \frac{\sqrt{15}}{4}.$$

Assim:

$$\tan\alpha = \frac{\operatorname{sen}\alpha}{\cos\alpha} = -\frac{2}{3} \times \left(-\frac{3}{\sqrt{5}}\right) = +\frac{2}{\sqrt{5}};$$

$$\tan\beta = \frac{\operatorname{sen}\beta}{\cos\beta} = \frac{1}{4} \times \frac{4}{\sqrt{15}} = \frac{1}{\sqrt{15}}$$

Logo:

$$\tan(\alpha+\beta) = \frac{\frac{2}{\sqrt{5}} + \frac{1}{\sqrt{15}}}{1 - \frac{2}{\sqrt{5}}\cdot\frac{1}{\sqrt{15}}} = \frac{\frac{2}{\sqrt{5}} + \frac{1}{\sqrt{3}\cdot\sqrt{5}}}{1 - \frac{2}{\sqrt{5}\cdot\sqrt{3}\cdot\sqrt{5}}} =$$

$$= \frac{\frac{1}{\sqrt{5}}\left(2+\frac{1}{\sqrt{3}}\right)}{1-\frac{2}{5\sqrt{3}}} = \frac{\frac{1}{\sqrt{5}}\cdot\left(\frac{2\sqrt{3}+1}{\sqrt{3}}\right)}{\frac{5\sqrt{3}-2}{5\sqrt{3}}}$$

$$= \frac{1+2\sqrt{3}}{5\sqrt{5}(5\sqrt{3}-2)}$$

(4) (b) $y = \frac{2\pi}{5} - 3 \operatorname{arcsen}(1-x)$

$$\Rightarrow y - \frac{2\pi}{5} = -3 \operatorname{arcsen}(1-x)$$

$$\Rightarrow \frac{\frac{2\pi}{5} - y}{3} = \operatorname{arcsen}(1-x) \qquad (*)$$

Domínio e imagem: note que $-1 \leq 1-x \leq 1 \iff 0 \leq x \leq 2$

Logo, $D(f) = [0, 2]$

Ainda, pela def. padrão de arcsen:

$$-\frac{\pi}{2} \leq \frac{\frac{2\pi}{5} - y}{3} \leq \frac{\pi}{2} \iff -\frac{3\pi}{2} \leq \frac{2\pi}{5} - y \leq \frac{3\pi}{2} \iff$$

$$\iff -\frac{3\pi}{2} - \frac{2\pi}{5} \leq -y \leq \frac{3\pi}{2} - \frac{2\pi}{5} \iff \cdots \iff -\frac{11\pi}{10} \leq y \leq \frac{19\pi}{10}$$

Logo, $Im(f) = [-\frac{11\pi}{10}, \frac{19\pi}{10}]$.

Continuando de $(*)$, temos:

$$\operatorname{sen}\left(\frac{2\pi}{15} - \frac{y}{3}\right) = 1 - x \Rightarrow x = 1 - \operatorname{sen}\left(\frac{2\pi}{15} - \frac{y}{3}\right)$$

Vamos considerar primeiramente, a construção gráfica de

$$y = 1 - \operatorname{sen}\left(\frac{2\pi}{15} - \frac{t}{3}\right).$$

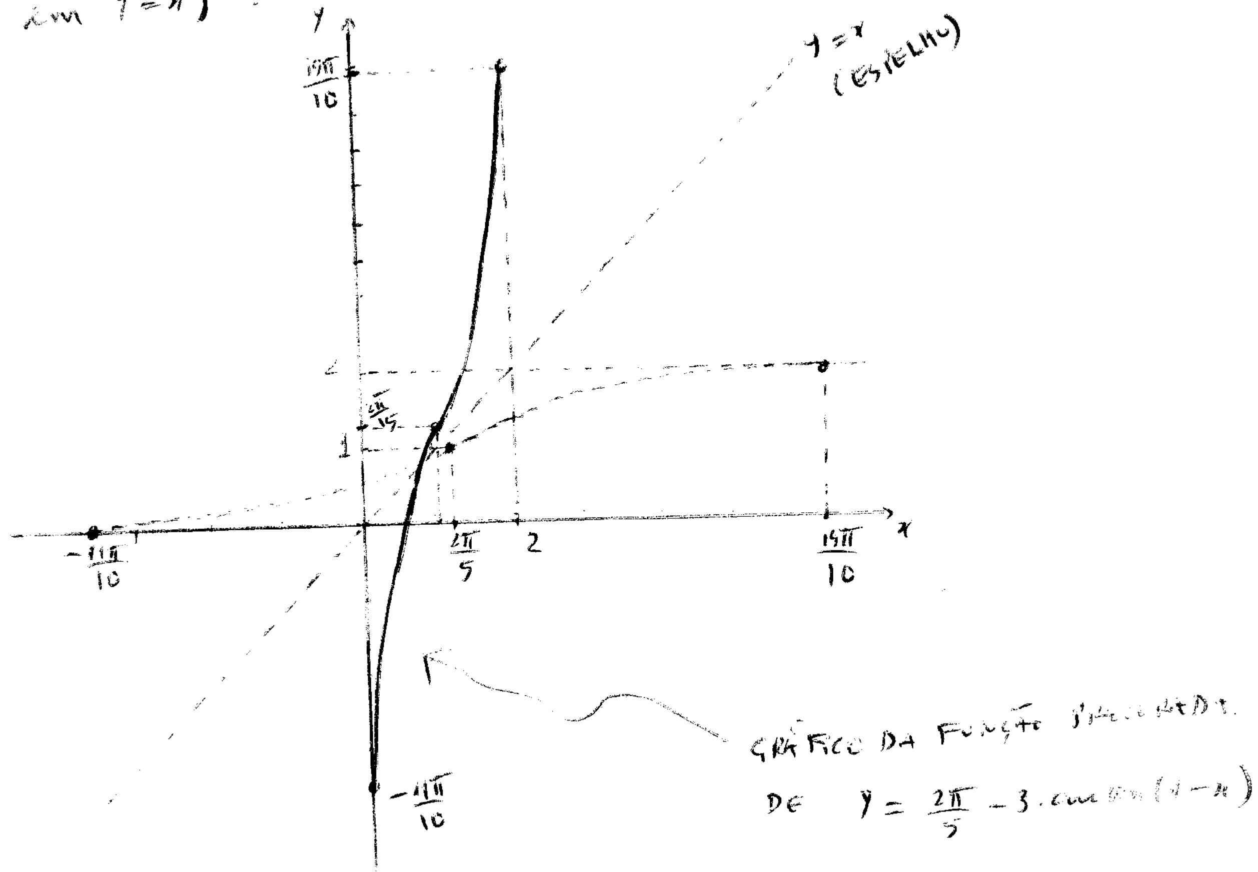
Então $t = \frac{2\pi}{15} - \frac{x}{3}$. Então obtemos $x = \frac{2\pi}{5} - 3t$ e daí:

$y = 1 - \operatorname{sen} t$

| $t$ | $y = 1 - \operatorname{sen} t$ | $x = \frac{2\pi}{5} - 3t$ |
|---|---|---|
| $-\frac{\pi}{2}$ | 2 | $\frac{2\pi}{5} + \frac{3\pi}{2} = \frac{19\pi}{10}$ |
| 0 | 1 | $\frac{2\pi}{5}$ |
| $\frac{\pi}{2}$ | 0 | $\frac{2\pi}{5} - \frac{3\pi}{2} = -\frac{11\pi}{10}$ |

(segue...) $\Rightarrow$

Esboço gráfico: (e uso da técnica de "espelhamento" em $y=x$):




07) Dados $\alpha = \arcsin \frac{1}{\sqrt{5}} \Rightarrow \operatorname{sen} \alpha = \frac{1}{\sqrt{5}}$ ; $\alpha \in 1.^o q$

$\beta = \arcsin \frac{2}{\sqrt{5}} \Rightarrow \operatorname{sen} \beta = \frac{2}{\sqrt{5}}$ ; $\beta \in 1.^o q$.

Queremos mostrar que $\alpha + \beta = \frac{\pi}{2}$. Isso equivale a mostrar que $\operatorname{sen}(\alpha+\beta) = \operatorname{sen}\frac{\pi}{2} = 1$.

De fato:

$$\begin{aligned}
\operatorname{sen}(\alpha+\beta) &= \operatorname{sen}\alpha \cdot \cos\beta + \operatorname{sen}\beta \cos\alpha \\
&= \operatorname{sen}\alpha \cdot \left(+\sqrt{1-\operatorname{sen}^2\beta}\right) + \operatorname{sen}\beta \cdot \left(+\sqrt{1-\operatorname{sen}^2\alpha}\right) \\
&= \frac{1}{\sqrt{5}} \cdot \left(\sqrt{1-\frac{4}{5}}\right) + \frac{2}{\sqrt{5}} \cdot \left(\sqrt{1-\frac{1}{5}}\right) \\
&= \frac{1}{\sqrt{5}} \cdot \frac{1}{\sqrt{5}} + \frac{2}{\sqrt{5}} \cdot \frac{2}{\sqrt{5}} = \frac{1}{5} + \frac{4}{5} = 1
\end{aligned}$$

$\square$.

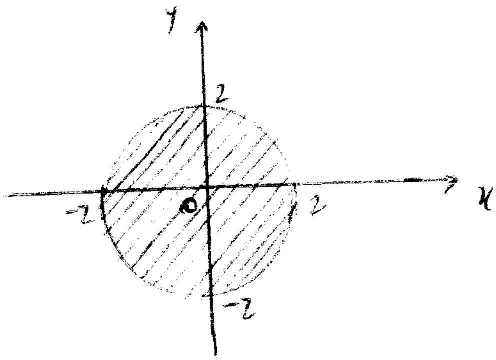
Lista 09.

10) (a) $|z| \leq 2$. $z = x + yi$

$|z| = \sqrt{x^2 + y^2}$.

$|z| \leq 2 \iff \sqrt{x^2+y^2} \leq 2 \iff x^2 + y^2 \leq 4$

(REGIÃO INTERNA DE UMA CIRCUNFERÊNCIA CENTRADA NA ORIGEM E RAIO 2)




INCLUI A "BORDA" POIS A DESIGUALDADE É "≤".

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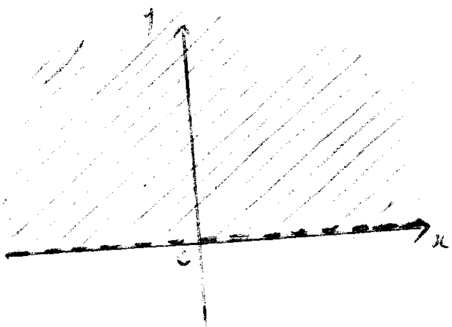
(b) $\operatorname{Im} z > 0$. $z = x + yi$

$\operatorname{Im}(z) = y > 0$.




SEMIPLANO SUPERIOR (onde $y > 0$)

NOTE QUE $\underline{\underline{y > 0}}$, POR ISSO O EIXO X FICA TRACEJADO, POIS NENHUM ELEMENTO PERTENCE AO CONJUNTO

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(c) $\operatorname{Re}\left(\frac{z-1}{z+1}\right) \leq 1$. Note que:

$$\frac{x + yi - 1}{x + yi + 1} = \frac{(x-1) + yi}{(x+1) + yi} \cdot \frac{(x+1) - yi}{(x+1) - yi} =$$

obs! FALTOU INSERIR "Re" !!!

~~$(x+1)(x+y)i$~~

(segue) $\Longrightarrow$

10

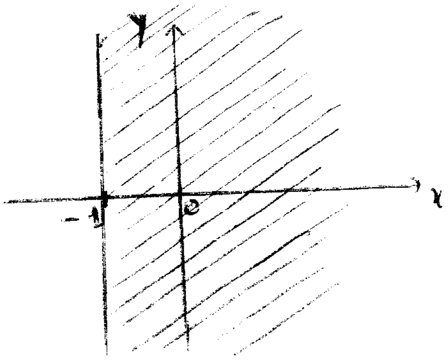
$$= \frac{((x-1)+yi)((x+1)-yi)}{(x+1)^2 - y^2 i^2} = \frac{x^2-1 - (x-1)yi + (x+1)yi + y^2}{(x+1)^2+y^2}$$

$$= \frac{x^2+y^2-1}{(x+1)^2+y^2} + \frac{(x+1)\cdot y - (x-1)y}{(x+1)^2+y^2}\, i \quad . \text{ Assim,}$$

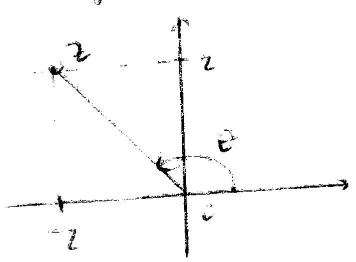
$$Re\left(\frac{z-1}{z+1}\right) \leq 1 \iff \frac{x^2+y^2-1}{(x+1)^2+y^2} \leq 1 \iff$$

$$\iff x^2+y^2-1 \leq x^2+2x+1+y^2 \iff 2x+1 \geq -1 \iff$$

$$\iff 2x \geq -2 \iff \boxed{x \geq -1}$$




12) (h) $\sqrt[5]{-2+2i}$. Dado $z = -2+2i$, denotando-o na forma trigonométrica, vamos obter:




$\theta \in 2^o Q \qquad \tan\theta = \frac{b}{a} = \frac{2}{-2} = -1$

$\theta = 135^\circ$

$\rho = \sqrt{a^2+b^2} = \sqrt{(-2)^2+(2)^2} = 2$

Logo: $z = \rho(\cos\theta + i\sin\theta)$

$\Rightarrow z = 2(\cos 135^\circ + i \operatorname{sen} 135^\circ)$

(segue...)

Queremos determinar as raízes quintas de z, ou seja,

$$\sqrt[5]{z} = \underbrace{\sqrt[5]{\rho}}_{=\sqrt[5]{2}} \cdot \left(\cos \frac{k\cdot 360^\circ + 135^\circ}{5} + i\ \text{sen}\ \frac{k\cdot 360^\circ + 135^\circ}{5}\right)$$

$k = 0:\quad w_0 = \sqrt[5]{2}\cdot\left(\cos \frac{135^\circ}{5} + i\ \text{sen}\frac{135^\circ}{5}\right)$

$\quad = \sqrt[5]{2}\cdot\left(\cos 27^\circ + i\ \text{sen}\ 27^\circ\right)$

$k = 1:\quad w_1 = \sqrt[5]{2}\cdot\left(\cos \frac{495^\circ}{5} + i\ \text{sen}\ \frac{495^\circ}{5}\right) = \sqrt[5]{2}\cdot\left(\cos 99^\circ + i\ \text{sen}\ 99^\circ\right)$

$k = 2:\quad w_2 = \sqrt[5]{2}\cdot\left(\cos 171^\circ + i\ \text{sen}\ 171^\circ\right)$

$k = 3:\quad w_3 = \sqrt[5]{2}\ \left(\cos 243^\circ + i\ \text{sen}\ 243^\circ\right)$

$k = 4:\quad w_4 = \sqrt[5]{2}\cdot\left(\cos 315^\circ + i\ \text{sen}\ 315^\circ\right)$

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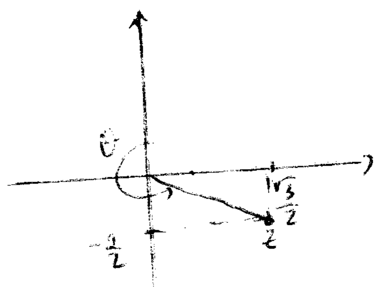
13) $z = \frac{2}{\sqrt{3}+i}\cdot\frac{\sqrt{3}-i}{\sqrt{3}-i} = \frac{2\,(\sqrt{3}-i)}{3+1} = \frac{\sqrt{3}}{2} - \frac{1}{2}i$




$\theta \in 4^{\circ}q.$ $\quad \tan\theta = \frac{b}{a} = \frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}} = -\frac{\sqrt{3}}{3}$

$\theta = 360^\circ - 30^\circ = 330^\circ$

$\rho = \sqrt{a^2+b^2} = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = 1$

Logo: $z = \rho\,(\cos\theta + i\ \text{sen}\ \theta)$

$z = 1\,(\cos 330^\circ + i\ \text{sen}\ 330^\circ).$