ÁLGEBRA LINEAR – RESOLUÇÃO DE EXERCÍCIOS DA LISTA 02.
PROF. MAURÍCIO ZAHN.

01) Basta, por exemplo, tomar $A_{2\times 2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ e $B_{2\times 2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Então, $A \neq 0$ e $B \neq 0$, mas

$$A \cdot B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0_{2\times 2}.$$

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02) Sejam $A_{n\times n}$ e $B_{n\times n}$ triangulares superiores e $C = A \cdot B$.

Então

$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}, \quad \text{onde} \begin{cases} a_{ij} = 0, \text{ se } i > j \\ \text{e} \\ b_{ij} = 0, \text{ se } i > j \end{cases}$$

(Pois $A$ e $B$ são triang. sup.)

Para mostrar que $C = A \cdot B$ é triangular superior, precisamos mostrar que $c_{ij} = 0$ se $i > j$. De fato:

$$c_{21} = \sum_{k=1}^{n} a_{2k} \cdot b_{k1} = \underbrace{a_{21} \cdot b_{11}}_{\substack{= 0 \text{ pois} \\ a_{21} = 0}} + \underbrace{a_{22} \cdot b_{21}}_{\substack{= 0, \text{ pois} \\ b_{21} = 0}} + \underbrace{a_{23} \cdot b_{31}}_{\substack{= 0, \text{ pois} \\ b_{31} = 0}} + \cdots + \underbrace{a_{2n} \cdot b_{n1}}_{\substack{= 0, \text{ pois} \\ b_{n1} = 0}} = 0$$

$$c_{31} = \sum_{k=1}^{n} a_{3k} \, b_{k1} = \underbrace{a_{31} \cdot b_{11}}_{\substack{= 0, \text{ pois} \\ a_{31} = 0}} + \underbrace{a_{32} \cdot b_{21}}_{\substack{= 0, \text{ pois} \\ a_{32} = 0}} + \underbrace{a_{33} \cdot b_{31}}_{\substack{= 0, \text{ pois} \\ b_{31} = 0}} + \cdots + \underbrace{a_{3n} \cdot b_{n1}}_{\substack{= 0, \text{ pois} \\ b_{n1} = 0}} = 0$$

$$\underset{\sim}{c_{32}} = \sum_{k=1}^{n} a_{3k} \cdot b_{k2} = \underbrace{a_{31} \cdot b_{12}}_{\overset{\shortparallel}{0},\ \text{pois } a_{31}=0} + \underbrace{a_{32} \cdot b_{22}}_{\overset{\shortparallel}{0},\ \text{pois } a_{32}=0} + \underbrace{a_{33} \cdot b_{32}}_{\overset{\shortparallel}{0},\ \text{pois } a_{32}=0} + \cdots + \underbrace{a_{3n} \cdot b_{n2}}_{\overset{\shortparallel}{0},\ \text{pois } b_{n2}=0} = \underset{\sim}{0};$$

e assim por diante, onde a diagonal fica

$$\underset{\sim}{c_{11}} = \sum_{k=1}^{n} a_{1k} \cdot b_{k1} = \underbrace{a_{11} \cdot b_{11}}_{\neq 0} + \underbrace{a_{12} \cdot b_{21}}_{=0} + \cdots + \underbrace{a_{1n} \cdot b_{n1}}_{=0} = \underset{\sim}{a_{11} \cdot b_{11}}$$

$$\underset{\sim}{c_{22}} = \sum_{k=1}^{n} a_{2k}\, b_{k2} = \underbrace{a_{21} \cdot b_{12}}_{\overset{\shortparallel}{0}} + a_{22} \cdot b_{22} + \underbrace{a_{23} \cdot b_{32}}_{\overset{\shortparallel}{0}} + \cdots + \underbrace{a_{2n} \cdot b_{n2}}_{\overset{\shortparallel}{0}} =$$

$$= \underset{\sim}{a_{22} \cdot b_{22}};$$

em geral:

$$\underset{\sim}{c_{\ell\ell}} = \sum_{k=1}^{n} a_{\ell k}\, b_{k\ell} = \underbrace{a_{\ell 1} \cdot b_{1\ell}}_{\overset{\shortparallel}{0}} + \cdots + \underbrace{a_{\ell\ell} \cdot b_{\ell\ell}} + \underbrace{a_{\ell,\ell+1}\, b_{\ell+1,\ell}}_{\overset{\shortparallel}{0}} + \cdots + \underbrace{a_{\ell n}\, b_{n\ell}}_{\overset{\shortparallel}{0}}$$

$$= \underset{\sim}{a_{\ell\ell} \cdot b_{\ell\ell}}.$$

Portanto, $C = A \cdot B$ é tal que $c_{ij} = 0$ se $i > j$; ou seja, é triangular superior, com diagonal

$$a_{11} \cdot b_{11},\ a_{22} \cdot b_{22}, \ldots, a_{\ell\ell} \cdot b_{\ell\ell}, \ldots, a_{nn} \cdot b_{nn}.$$

04) Suponha $A$ inversível. Logo, existe $A^{-1}$ inversa de $A$ tal que $A \cdot A^{-1} = I_n$ e $A^{-1} \cdot A = I_n$.

Mostrar: $A^t$ é inversível.

Note que

$$\left.\begin{array}{l} (A \cdot A^{-1})^t = I^t = I \ , \\ \quad \| \\ (A^{-1})^t \cdot A^t \end{array}\right\} \Rightarrow \boxed{(A^{-1})^t \cdot A^t = I} \quad (*)$$

e que:

$$\left.\begin{array}{l} (A^{-1} \cdot A)^t = I^t = I \\ \quad \| \\ A^t \cdot (A^{-1})^t \end{array}\right\} \Rightarrow \boxed{A^t \cdot (A^{-1})^t = I} \quad (**)$$

Logo, de $(*)$ e $(**)$ segue que $A^t$ é também inversível, com inversa $(A^t)^{-1} = (A^{-1})^t$. $\square$

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05) Suponha $A$ inversível e que $A \cdot B = A \cdot C$.

Mostrar: $B = C$ ("lei do corte")

Sendo $A$ inversível, $\exists\, A^{-1}$ inversa de $A$. Assim:

$$A \cdot B = A \cdot C$$

$$\Rightarrow A^{-1} \cdot (A \cdot B) = A^{-1} \cdot (A \cdot C) \rightsquigarrow \text{(multiplicando à esquerda por } A^{-1}\text{)}.$$

$$\Rightarrow \underbrace{(A^{-1} \cdot A)}_{I} \cdot B = \underbrace{(A^{-1} \cdot A)}_{I} \cdot C \Rightarrow \boxed{B = C}$$

$\square$

10) (b).

$$B = \begin{pmatrix} 5 & -9 & 6 \\ 0 & 2 & 3 \\ 0 & 0 & 7 \end{pmatrix} \xrightarrow{l_1 \hookrightarrow \frac{1}{5} l_1} \begin{pmatrix} 1 & -\frac{9}{5} & \frac{6}{5} \\ 0 & 2 & 3 \\ 0 & 0 & 7 \end{pmatrix} \xrightarrow{l_2 \hookrightarrow \frac{1}{2} l_2}$$

$$\begin{pmatrix} 1 & -\frac{9}{5} & \frac{6}{5} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 7 \end{pmatrix} \xrightarrow{l_3 \hookrightarrow \frac{1}{7} l_3} \begin{pmatrix} 1 & -\frac{9}{5} & \frac{6}{5} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{l_2 \hookrightarrow l_2 - \frac{3}{2} l_3}$$

$$\begin{pmatrix} 1 & -\frac{9}{5} & \frac{6}{5} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{l_1 \hookrightarrow l_1 + \frac{9}{5} l_2} \begin{pmatrix} 1 & 0 & \frac{6}{5} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{l_1 \hookrightarrow l_1 - \frac{6}{5} l_3}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Ou seja, a forma escalonada reduzida por linhas da matriz B é a identidade.
(isso aconteceu porque B é invertível)

11) $\begin{cases} x + y + z = 6 \\ x - y + 2z = 2 \\ 2x + y - z = 3 \end{cases}$

(a) A matriz aumentada do sistema é:

$$\begin{pmatrix} 1 & 1 & 1 & 6 \\ 1 & -1 & 2 & 2 \\ 2 & 1 & -1 & 3 \end{pmatrix} \xrightarrow{l_2 \leftarrow l_2 - l_1} \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -4 \\ 2 & 1 & -1 & 3 \end{pmatrix}$$

$$\xrightarrow{l_3 \leftarrow l_3 - 2l_1} \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -4 \\ 0 & -1 & -3 & -9 \end{pmatrix} \xrightarrow{l_2 \leftrightarrow l_3} \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & -1 & -3 & -9 \\ 0 & -2 & 1 & -4 \end{pmatrix}$$

$$\xrightarrow{l_2 \leftarrow -1 \cdot l_2} \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 9 \\ 0 & -2 & 1 & -4 \end{pmatrix} \xrightarrow{l_3 \leftarrow l_3 + 2l_2} \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 9 \\ 0 & 0 & 7 & 14 \end{pmatrix}$$

$$\xrightarrow{l_3 \leftarrow \frac{1}{7} l_3} \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 9 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

Logo: $\boxed{z = 2}$ ;

$y + 3z = 9$

$\Rightarrow y + 3 \cdot (2) = 9 \Rightarrow \boxed{y = 3}$ ;

e $x + y + z = 6$

$\Rightarrow x + 3 + 2 = 6 \Rightarrow \boxed{x = 1}$

Solução: $x = 1$; $y = 3$ e $z = 2$., ou ainda;

$S = \{(x, y, z) = (1, 3, 2)\}$.

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(b) $\quad A\,x = b \iff \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 2 & 1 & -1 \end{pmatrix}}_{A} \cdot \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix}}_{b}$

A solução será, sendo A invertível:

$$\underbrace{A^{-1} \cdot A}_{I}\, X = A^{-1} \cdot b \implies X = A^{-1} \cdot b \,;$$

onde $(A \,\vdots\, I) = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 2 & 0 & 1 & 0 \\ 2 & 1 & -1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{l_2 \hookrightarrow l_2 - l_1}$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & -1 & 1 & 0 \\ 2 & 1 & -1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{l_3 \hookrightarrow l_3 - 2l_1} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & -1 & 1 & 0 \\ 0 & -1 & -3 & -2 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{l_2 \leftrightarrow l_3} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -3 & -2 & 0 & 1 \\ 0 & -2 & 1 & -1 & 1 & 0 \end{array}\right) \xrightarrow{l_2 \hookrightarrow -1 \cdot l_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 2 & 0 & -1 \\ 0 & -2 & 1 & -1 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{l_3 \hookrightarrow l_3 + 2l_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 2 & 0 & -1 \\ 0 & 0 & 7 & 3 & 1 & -2 \end{array}\right) \xrightarrow{l_3 \hookrightarrow \frac{1}{7} l_3} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 2 & 0 & -1 \\ 0 & 0 & 1 & \frac{3}{7} & \frac{1}{7} & -\frac{2}{7} \end{array}\right)$$

$$\xrightarrow{l_2 \hookrightarrow l_2 - 3l_3} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{5}{7} & -\frac{3}{7} & -\frac{1}{7} \\ 0 & 0 & 1 & \frac{3}{7} & \frac{1}{7} & -\frac{2}{7} \end{array}\right) \xrightarrow{l_1 \hookrightarrow l_1 - l_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{2}{7} & +\frac{3}{7} & +\frac{1}{7} \\ 0 & 1 & 0 & \frac{5}{7} & -\frac{3}{7} & -\frac{1}{7} \\ 0 & 0 & 1 & \frac{3}{7} & \frac{1}{7} & -\frac{2}{7} \end{array}\right)$$

$$\xrightarrow{l_1 \hookrightarrow l_1 - l_3} \Big( \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{= I} \;\Big|\; \underbrace{\begin{matrix} -\frac{1}{7} & \frac{2}{7} & \frac{3}{7} \\ \frac{5}{7} & -\frac{3}{7} & -\frac{1}{7} \\ \frac{3}{7} & \frac{1}{7} & -\frac{2}{7} \end{matrix}}_{= A^{-1}} \Big) \Longrightarrow A^{-1} = \begin{pmatrix} -\frac{1}{7} & \frac{2}{7} & \frac{3}{7} \\ \frac{5}{7} & -\frac{3}{7} & -\frac{1}{7} \\ \frac{3}{7} & \frac{1}{7} & -\frac{2}{7} \end{pmatrix}$$

Logo, a solução $X = A^{-1} \cdot b$ ficará:

$$X = \begin{pmatrix} -\frac{1}{7} & \frac{2}{7} & \frac{3}{7} \\ \frac{5}{7} & -\frac{3}{7} & -\frac{1}{7} \\ \frac{3}{7} & \frac{1}{7} & -\frac{2}{7} \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -\frac{6}{7} + \frac{4}{7} + \frac{9}{7} \\ \frac{30}{7} - \frac{6}{7} - \frac{3}{7} \\ \frac{18}{7} + \frac{2}{7} - \frac{6}{7} \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix};$$

ou seja, a solução será $x = 1$, $y = 3$ e $z = 2$, a mesma encontrada no item anterior.

12) $\begin{cases} x + y - z = 0 \\ 2x + 4y - z = 0 \\ 3x + 2y + 2z = 0 \end{cases}$ (obs. ARRUMAR ESSA ÚLTIMA EQUAÇÃO NO ENUNCIADO!)

SOLUÇÃO

$$\begin{pmatrix} 1 & 1 & -1 & 0 \\ 2 & 4 & -1 & 0 \\ 3 & 2 & 2 & 0 \end{pmatrix} \xrightarrow{l_2 \hookrightarrow l_2 - 2l_1} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 \\ 3 & 2 & 2 & 0 \end{pmatrix}$$

$$\xrightarrow{l_3 \hookrightarrow l_3 - 3l_1} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & -1 & 5 & 0 \end{pmatrix} \xrightarrow{l_2 \leftrightarrow l_3} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & -1 & 5 & 0 \\ 0 & 2 & 1 & 0 \end{pmatrix}$$

$$\xrightarrow{l_2 \hookrightarrow -1 . l_2} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 2 & 1 & 0 \end{pmatrix} \xrightarrow{l_3 \hookrightarrow l_3 - 2l_2} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 11 & 0 \end{pmatrix}$$

$$\Rightarrow 11z = 0 \Rightarrow z = 0$$

$$y - 5z = 0 \Rightarrow y - 0 = 0 \Rightarrow y = 0$$

$$x + \underset{0}{\underset{\shortparallel}{y}} - \underset{0}{\underset{\shortparallel}{z}} = 0 \Rightarrow x = 0$$

Portanto, tal sistema só possui a solução $(x, y, z) = (0, 0, 0)$.

13) Apenas as respostas finais (feito no MAPLE):

(a) $S = \{(x, y, z) = (2, -1, 1)\}$.

(b) o sistema não possui solução; é incompatível.

(c) $S = \{(x, y, z) = (-3-t\,;\, 2+2t\,;\, t)\,;\, t \in \mathbb{R}\}$.

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15) Tirando o m.m.c $(x^2+2x+2\,;\, x-1\,;\, x+1)$, temos:

$$\frac{3x^3+4x^2-6x}{(x^2+2x+2)(x+1)(x-1)} \equiv \frac{(x^2-1)\cdot(Ax+B)+C(x^2+2x+2)\cdot(x+1)+(\bigstar)}{(x^2+2x+2)(x+1)(x-1)}$$

onde:

$(\bigstar) = D\cdot(x^2+2x+2)(x-1)$

(escrevi assim pois não teve espaço na escrita acima)

Logo:

$$3x^3+4x^2-6x \equiv Ax^3+Bx^2-Ax-B+Cx^3+Cx^2+2Cx^2+2Cx+2Cx+2C+$$
$$+Dx^3-Dx^2+2Dx^2-\cancel{2Dx}+\cancel{2Dx}-2D$$

Como há uma igualdade entre polinômios, segue que deve haver uma igualdade entre os coeficientes com mesmo expoente nas incógnitas. Assim, vamos obter:

$$\begin{cases} A + C + D = 3 \\ B + 3C + D = 4 \\ -A + 4C = -6 \\ -B + 2C - 2D = 0 \end{cases} \iff \begin{pmatrix} 1 & 0 & 1 & 1 & 3 \\ 0 & 1 & 3 & 1 & 4 \\ -1 & 0 & 4 & 0 & -6 \\ 0 & -1 & 2 & -2 & 0 \end{pmatrix}$$

(MATRIZ AUMENTADA)

et cetera ...