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On the isomorphic classification of $C(K,X)$ spaces

Elói Medina Galego [a], Maurício Zahn [b]

[a] University of São Paulo, Department of Mathematics, IME, Rua do Matão 1010, São Paulo, Brazil
[b] Federal University of Pelotas, Department of Mathematics and Statistics, IFM, Campus Universitário Capão do Leão, s/n, Rio Grande do Sul, Brazil

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ABSTRACT

We provide isomorphic classifications of some $C(K,X)$ spaces, the Banach spaces of all continuous X-valued functions defined on infinite compact metric spaces K, equipped with the supremum norm. We first introduce the concept of ω_1-quotient of Banach spaces X. Thus, we prove that if X has some ω_1-quotient which is uniformly convex, then for all K_1 and K_2 the following statements are equivalent:

(a) $C(K_1,X)$ is isomorphic to $C(K_2,X)$.
(b) $C(K_1)$ is isomorphic to $C(K_2)$.

This allows us to classify, up to an isomorphism, some $C(K, Y \oplus l_p(\Gamma))$ spaces, $1 < p \leq \infty$, and certain $C(S)$ spaces involving large compact Hausdorff spaces S.

1. Introduction

We refer the reader to [1,7,18] for details on standard notation and terminology we use in the paper. For a compact Hausdorff topological space K let $C(K,X)$ denote the Banach space of all continuous X-valued functions defined on K, equipped with the supremum norm. This space will be denoted by $C(K)$ in the case where $X = \mathbb{R}$. As usual, in the case where K is the interval of ordinals $[0,\alpha]$ endowed with the order topology, these spaces will be denoted respectively by $C(\alpha, X)$ and $C(\alpha)$. When α is the first infinite ordinal, these spaces will be also denoted by $c_0(X)$ and c_0 respectively. If K and S are compact Hausdorff spaces, we denote by $K \oplus S$ and $K \times S$ respectively the topological sum and the topological product of K and S. For a fixed cardinal number $\mathfrak{m} \geq 1$, $\mathbf{2}^{\mathfrak{m}}$ denotes the Cantor cube, that is, the product of $\mathfrak{m}$ family of copies of the two-point space $\mathbf{2}$, provided with the product topology. If X and Y are Banach spaces, then $X \sim Y$ means that X is isomorphic to Y and $X \twoheadrightarrow Y$ means that Y is isomorphic to a quotient of X. Finally, the symbol $X \oplus Y$ denotes the Cartesian product of X and Y.

E-mail addresses: eloi@ime.usp.br (E.M. Galego), mauricio.zahn@ufpel.edu.br (M. Zahn).

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The central result on the isomorphic classification of separable $C(K)$ spaces, that is, K are metric spaces, is Milutin's Theorem [13], see [16]. This result states that if K is an uncountable compact metric space, then

$$C(K) \sim C(\mathbf{2}^{\aleph_0}). \tag{1.1}$$

In the case where K is a countable compact metric space, a classical Mazurkiewicz and Sierpiński's Theorem [12] asserts that K is homeomorphic to some interval of ordinals $[0, \alpha]$ for some ordinal $\alpha < \omega_1$, where ω_1 is the first uncountable ordinal. The isomorphic classification of the $C(\alpha)$ spaces was done by Bessaga and Pełczyński [2] in the following way. Let ξ and η be two ordinals such that $\omega \leq \xi \leq \eta < \omega_1$. Then

$$C(\xi) \sim C(\eta) \Leftrightarrow \eta < \xi^{\omega}. \tag{1.2}$$

In the present paper we are mainly interested in getting the isomorphic classification of certain spaces involving the spaces (1.1) and (1.2). The starting point of our research is the fact that recently in [10] it was provided an extension of (1.2) to the vector-valued case. Namely, recall that a subspace H of a Banach space X is a maximal factor of X whenever X is the direct sum of H and some subspace Y of X such that every finite sum Y^n of Y contains no copy of H. Then, the main result of [10] is as follows.

Theorem 1.1. *Let X be a Banach space containing some uniformly convex maximal factor and ordinals $\omega \leq \xi \leq \eta < \omega_1$. Then*

$$C(\xi, X) \sim C(\eta, X) \Leftrightarrow \eta < \xi^{\omega}.$$

Of course Theorem 1.1 can be applied to obtain the isomorphic classifications of so many $C(\alpha, X)$ spaces, where $\omega \leq \alpha < \omega_1$. In particular, since $C(\mathbf{2}^{\mathfrak{m}})$ contains no copy of the classical uniformly convex Banach spaces $l_p(\Gamma)$, $1 < p < \infty$, whenever Γ is an *uncountable* set [5], [14, Proposition 8.11] and moreover

$$C(\alpha, C(\mathbf{2}^{\mathfrak{m}})) \sim C(\mathbf{2}^{\mathfrak{m}}), \tag{1.3}$$

for all $\omega \leq \alpha < \omega_1$ and infinite cardinal $\mathfrak{m}$, it follows by Theorem 1.1 that the isomorphic classification of the following spaces is the same as that of $C(\alpha)$ spaces, $\omega \leq \alpha < \omega_1$, mentioned in (1.2)

$$C(\alpha, C(\mathbf{2}^{\mathfrak{m}}) \oplus l_p(\Gamma)) \sim C(\mathbf{2}^{\mathfrak{m}}) \oplus C(\alpha, l_p(\Gamma)). \tag{1.4}$$

On the other hand, observe that when Γ is finite, the spaces (1.4) are isomorphic to $C(\mathbf{2}^{\mathfrak{m}})$, for all $\omega \leq \alpha < \omega_1$ and infinite cardinal $\mathfrak{m}$.

Then, it is natural to look for the complete isomorphic classification of the spaces (1.4) when $1 \leq p \leq \infty$. The study of this question in the case where $p \neq 1$ led us to obtain two more general isomorphic classifications of some $C(K, X)$ spaces for infinite compact metric spaces K. So, our contribution to answering the above question will be presented as a consequence of them. More precisely, in Section 3 we will prove:

Theorem 1.2. *Let Y be a Banach space, $1 < p < \infty$ and Γ be an infinite set. Suppose that Y^* contains no copy of l_q, where $1/p + 1/q = 1$. Then for all infinite compact metric spaces K_1 and K_2,*

$$C(K_1, Y \oplus l_p(\Gamma)) \sim C(K_2, Y \oplus l_p(\Gamma)) \Leftrightarrow C(K_1) \sim C(K_2).$$

Therefore in the case where $1 < p < 2$, since the dual of each $C(\mathbf{2}^{\mathfrak{m}})$ space contains no copy of l_q, with $q > 2$ [1, Theorem 6.4.19.i], the isomorphic classification of the spaces (1.4) with $1 < p < 2$ is a corollary of

Theorem 1.2 regardless of whether the infinite set Γ is countable or uncountable. This furnishes a solution to [10, Problem 4.3.a] when $1 < p < 2$.

Furthermore, recall that the density character of a topological space F (denoted by $dens\, F$) is the smallest cardinality of a dense subset of F and denote by $|\Gamma|$ the cardinality of a set Γ. In Section 4 we will prove the following theorem.

Theorem 1.3. *Let Y be a Banach space and Γ an infinite set. Suppose that $dens\, Y < 2^{|\Gamma|}$. Then for all infinite compact metric spaces K_1 and K_2,*

$$C(K_1, Y \oplus l_\infty(\Gamma)) \sim C(K_2, Y \oplus l_\infty(\Gamma)) \Leftrightarrow C(K_1) \sim C(K_2).$$

Thus, since $dens\, C(\mathbf{2}^{\mathfrak{m}}) = \mathfrak{m}$, for every infinite cardinal $\mathfrak{m}$ [18, Corollary 8.2.6 and Proposition 7.6.5], Theorem 1.3 provides the isomorphic classification of the spaces (1.4) when $p = \infty$ and $\aleph_0 \leq \mathfrak{m} < 2^{|\Gamma|}$. In the case where $\mathfrak{m} = |\Gamma| = \aleph_0$, Theorem 1.3 solves [10, Problem 4.3.c].

In order to prove Theorems 1.2 and 1.3, in the next section we state our main result (Theorem 2.4) which is a suitable extension of Theorem 1.1.

2. The isomorphic classification of certain $C(K, X)$ spaces

Concerning Theorem 1.1 our main technical improvement in this paper is to replace the uniformly convex maximal factor of X by a similarly positioned subspace of X which has a uniformly convex quotient. We start by introducing the following definition:

Definition 2.1. We say that a Banach space Z is an ω_1-quotient of a Banach space X if there exist subspaces A and B of X such that

(a) $X = A \oplus B$,
(b) $B \twoheadrightarrow Z$,
(c) $C(\xi, A) \oplus B^n \not\twoheadrightarrow c_0(Z)$, for every $\omega \leq \xi < \omega_1$ and $1 \leq n < \omega$.

Remark 2.2. The above definition was inspired by the proof of [9, Theorem 2]. This result states that if F is the uniformly convex Banach space introduced by Figiel in [8] and $Z = F^*$, then for all ordinals $\omega \leq \xi \leq \eta < \omega_1$,

$$C(\xi, C(\mathbf{2}^{\aleph_0}) \oplus Z) \sim C(\eta, C(\mathbf{2}^{\aleph_0}) \oplus Z) \Leftrightarrow \eta < \xi^\omega.$$

In order to prove this, it was shown that for all $1 \leq n < \omega$,

$$C(\mathbf{2}^{\aleph_0}) \oplus Z^n \not\twoheadrightarrow c_0(Z). \tag{2.1}$$

Thus, we can see Definition 2.1 as a refinement of this technical obstruction to maps onto c_0 sums. Indeed, according to (1.3) and (2.1) we deduce that the dual of the Figiel space F is an ω_1-quotient of $C(\mathbf{2}^{\aleph_0}) \oplus F^*$.

Remark 2.3. Notice that ω_1-quotients of a Banach space X are in fact quotients of X; while l_1 is not an ω_1-quotient of itself. Moreover, any Banach space Z containing no quotient isomorphic to c_0 is an ω_1-quotient of itself. Indeed, if the item (c) of Definition 2.1 does not hold with $A = 0$ and $B = Z$, then

$$Z^n \twoheadrightarrow c_0(Z) \twoheadrightarrow c_0,$$

for some $1 \leq n < \omega$. Therefore by [17, Theorem 2] c_0 is isomorphic to a quotient of Z, which is an absurd. In particular, each uniformly convex space is an ω_1-quotient of itself.

The aim of this section is to prove the following isomorphic classification.

Theorem 2.4. *Let X be a Banach space having an ω_1-quotient which is uniformly convex. Then for all infinite compact metric spaces K_1 and K_2,*

$$C(K_1, X) \sim C(K_2, X) \Leftrightarrow C(K_1) \sim C(K_2).$$

Before proving this theorem, we shall state two propositions.

Proposition 2.5. *Let A, B and Z be Banach spaces such that Z is uniformly convex and ordinals $\omega \leq \xi \leq \eta < \omega_1$. Suppose that*

(a) $B \twoheadrightarrow Z$,
(b) $A \oplus B^n \not\twoheadrightarrow c_0(Z)$, *for every* $1 \leq n < \omega$.

Then

$$A \oplus C(\xi, B) \twoheadrightarrow C(\eta, Z) \Longrightarrow \eta < \xi^\omega.$$

Proof. First we will show by transfinite induction that for any $0 \leq \alpha < \omega_1$ and $\gamma < \omega^{\omega^\alpha}$

$$A \oplus C(\gamma, B) \not\twoheadrightarrow C(\omega^{\omega^\alpha}, Z). \tag{2.2}$$

The hypothesis (b) covers the case $\alpha = 0$. Next suppose that $\beta = \alpha + 1$, for some ordinal α, and for all $\gamma < \omega^{\omega^\alpha}$ (2.2) holds. Assume that

$$A \oplus C(\gamma_1, B) \twoheadrightarrow C(\omega^{\omega^\beta}, Z) = C((\omega^{\omega^\alpha})^\omega, Z), \tag{2.3}$$

for some $\gamma_1 < \omega^{\omega^\beta}$.

Now observe that if $\gamma_1 < \omega^{\omega^\alpha}$ then $C(\omega^{\omega^\alpha}, B) \twoheadrightarrow C(\gamma_1, B)$. Moreover, if $\omega^{\omega^\alpha} \leq \gamma_1$, then by (1.2) we have $C(\omega^{\omega^\alpha}, B) \sim C(\gamma_1, B)$. Thus, by (2.3)

$$A \oplus C(\omega^{\omega^\alpha}, B) \twoheadrightarrow C((\omega^{\omega^\alpha})^\omega, Z).$$

Therefore by [9, Proposition 5.4] there exists an ordinal $\gamma_2 < \omega^{\omega^\alpha}$ such that

$$A \oplus C(\gamma_2, B) \twoheadrightarrow C(\omega^{\omega^\alpha}, Z),$$

but this contradicts (2.2).

Finally suppose that β is a limit ordinal and for all $\alpha < \beta$ and $\gamma < \omega^{\omega^\alpha}$ (2.2) holds.

Assume that

$$A \oplus C(\gamma_1, B) \twoheadrightarrow C(\omega^{\omega^\beta}, Z), \tag{2.4}$$

for some $\gamma_1 < \omega^{\omega^\beta}$. Pick an ordinal α such that $\gamma_1 < \omega^{\omega^\alpha} < \omega^{\omega^\beta}$. According to (2.4)

$$A \oplus C(\gamma_1, B) \twoheadrightarrow C(\omega^{\omega^\alpha}, Z),$$

contradicting (2.2).

Now we pass to prove the statement of the proposition. Assume then that

$$A \oplus C(\xi, B) \twoheadrightarrow C(\eta, Z), \tag{2.5}$$

with $\omega \leq \xi \leq \eta < \omega_1$.

In view of (1.2) the spaces $C(\omega^{\omega^\gamma})$, for $0 \leq \gamma < \omega_1$, are a complete set of representatives of the isomorphism classes of $C(\xi)$ spaces for $0 \leq \xi < \omega_1$. So, let α be the ordinal such that

$$C(\eta) \sim C(\omega^{\omega^\alpha}).$$

Notice that $\eta < \omega^{\omega^{\alpha+1}}$ and

$$C(\eta, Z) \sim C(\omega^{\omega^\alpha}, Z). \tag{2.6}$$

According to (2.5) and (2.6)

$$A \oplus C(\xi, B) \twoheadrightarrow C(\omega^{\omega^\alpha}, Z). \tag{2.7}$$

Hence by (2.2) and (2.7) we have $\omega^{\omega^\alpha} \leq \xi$ and therefore $\omega^{\omega^{\alpha+1}} \leq \xi^\omega$. Consequently $\eta < \xi^\omega$. □

The following remark will be useful in the sequel.

Remark 2.6. Suppose that Z is isomorphic to a quotient of the Banach space B. It follows from the Bartle–Graves continuous selection for quotient maps [4, p. 52] that $C(\xi, Z)$ is isomorphic to a quotient of $C(\xi, B)$ for every ordinal ξ.

Proposition 2.7. *Let X be a Banach space having an ω_1-quotient which is uniformly convex. Then for all ordinals $\omega \leq \xi \leq \eta < \omega_1$,*

$$C(\xi, X) \sim C(\eta, X) \Rightarrow \eta < \xi^\omega.$$

Proof. By hypothesis there exist a uniformly convex space Z and subspaces A and B of X satisfying (a), (b) and (c) of Definition 2.1. First of all observe that if we fix an ordinal $\omega \leq \xi_0 < \omega_1$, since

$$C(\xi_0, A) \oplus B^n \not\twoheadrightarrow c_0(Z),$$

for every $1 \leq n < \omega$, it follows from Proposition 2.5 applied to the spaces $C(\xi_0, A)$, B and Z that for all ordinals $\omega \leq \xi \leq \eta < \omega_1$,

$$C(\xi_0, A) \oplus C(\xi, B) \twoheadrightarrow C(\eta, Z) \Longrightarrow \eta < \xi^\omega. \tag{2.8}$$

Now, pick ordinals $\omega \leq \xi \leq \eta < \omega_1$ and suppose that

$$C(\xi, X) \sim C(\eta, X). \tag{2.9}$$

Since $X = A \oplus B$ and $B \twoheadrightarrow Z$, by (2.9) and Remark 2.6 we have

$$C(\xi, A) \oplus C(\xi, B) \sim C(\eta, A) \oplus C(\eta, B) \twoheadrightarrow C(\eta, Z).$$

According to (2.8) with $\xi_0 = \xi$ we obtain $\eta < \xi^\omega$. □

Now we are ready to prove the main result of this paper.

Proof of Theorem 2.4. The condition is clearly sufficient. Let us show necessity. Suppose then that $C(K_1, X)$ is isomorphic to $C(K_2, X)$, for some infinite compact metric spaces K_1 and K_2. We distinguish two cases:

Case 1. K_1 and K_2 are countable. Let ξ and η be infinite countable ordinals such that $C(K_1)$ is isomorphic to $C(\xi)$ and $C(K_2)$ is isomorphic to $C(\eta)$. Hence

$$C(\xi, X) \sim C(\eta, X).$$

Without loss of generality we may assume that $\xi \leq \eta$. So, by Proposition 2.7 and (1.2) we infer that $C(K_1)$ is isomorphic to $C(K_2)$.

Case 2. K_2 is uncountable. In this case, by (1.1) it suffices to show that K_1 is also uncountable. Otherwise, there exists a countable ordinal ξ such that $C(K_1)$ is isomorphic to $C(\xi)$. Consequently,

$$C(\xi, X) \sim C(K_1, X). \tag{2.10}$$

Furthermore, it follows from (1.1) and (1.2) that

$$C([0, \xi^\omega] \times K_2) \sim C(K_2) \quad \text{and} \quad C(\xi^\omega) \sim C([0, \xi^\omega] \times [0, \xi]).$$

Therefore

$$C(\xi^\omega, X) \sim C(\xi^\omega, C(\xi, X)) \sim C(\xi^\omega, C(K_2, X)) \sim C(K_2, X). \tag{2.11}$$

Thus, by (2.10) and (2.11) we see that

$$C(\xi, X) \sim C(\xi^\omega, X),$$

which contradicts Proposition 2.7 and the theorem follows. □

3. On the isomorphic classification of $C(K, Y \oplus l_p(\Gamma))$ spaces, $1 < p < \infty$

The purpose of this section is to provide the proof of Theorem 1.2. We shall denote by $\{e_{i,j}\}_{i,j=1}^\infty$ the canonical basis of $l_1(l_q)$, i.e.,

$$\left\| \sum_{i,j=1}^{\infty} a_{i,j} e_{i,j} \right\| = \sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} |a_{i,j}|^q \right)^{1/q},$$

for all $\{a_{i,j}\}_{i,j=1}^\infty \subseteq \mathbb{R}$.

The next lemma is obtained by a gliding hump argument and a simple perturbation argument which are well-known [11, p. 77], but we include the proof for completeness.

Lemma 3.1. *Let X be a Banach space and $1 < q < \infty$. Let T be a linear operator from $l_1(l_q)$ to $X \oplus l_q$ and P the natural projection from $X \oplus l_q$ onto l_q. Then:*

(a) For all double sequences $\{\epsilon_{ij}\}_{i,j=1}^\infty$ of positive numbers there exist a double sequence $\{b_{ij}\}_{i,j=1}^\infty \subseteq l_q$ with pairwise disjoint finite supports and subsequences $N_j \subseteq \mathbb{N}$ such that denoting $N_j = \{[i,j]\}_{i=1}^\infty$,

$$\|PT(e_{[i,j],j}) - b_{ij}\| < \epsilon_{ij}, \tag{3.1}$$

for every $1 \le i, j < \omega$.

(b) If T *is an into isomorphism then there exist subsequences* $N_j = \{[i,j]\}_{i=1}^{\infty} \subseteq \mathbb{N}$, $1 \le j < \omega$, *and an isomorphism* $\widetilde{T}$ *from the span of* $\{e_{[i,j],j}\}_{i,j=1}^{\infty}$ *into* $X \oplus l_q$ *such that* $\{P\widetilde{T}(e_{[i,j],j})\}_{i,j=1}^{\infty}$ *is a double sequence in* l_q *with pairwise disjoint finite supports.*

Proof. (a) Define an order $\prec$ on $\mathbb{N} \times \mathbb{N}$ by $(i,j) \prec (k,l)$ if, and only if, $i+j < k+l$ or $i+j = k+l$ and $i < k$.

Assume we already found the initial segments of $N_j = \{[i,j]\}_{i=1}^{k_j}$ for $(i,j) \prec (i_0, j_0)$. We need to find $[i_0, j_0]$ and $b_{i_0 j_0}$. Since $\{e_{i,j_0}\}_{i=1}^{\infty}$ tends weakly to zero, for i_0 large enough $\|P_i T(e_{i_0,j_0})\| < \epsilon_{i_0,j_0}/2$, where P_i is the projection onto S_i, the finite union of the supports of $\{b_{ij}\}_{(i,j)\prec(i_0,j_0)}$.

Now, for all $1 \le n < \omega$, denote by R_n the natural projection of l_q given by $R_n(\{a_i\}_{i=1}^{\infty}) = (a_1, a_2, \ldots, a_n, 0, 0, \ldots)$. Pick $1 \le m < \omega$ strictly greater than the maximum of S_i and such that

$$\|PT(e_{i_0,j_0}) - R_m PT(e_{i_0,j_0})\| < \epsilon_{i_0,j_0}/2.$$

So, it suffices to define $b_{i_0 j_0} = (R_m - P_i)PT(e_{i_0,j_0})$.

(b) Fix a double sequence $\{\epsilon_{ij}\}_{i,j=1}^{\infty}$ of positive numbers such that $\sum_{i,j=1}^{\infty} \epsilon_{ij}^p < 1/\|T^{-1}\|^p$, where $1/p + 1/q = 1$. By the item (a) there exist subsequences $N_j = \{[i,j]\}_{i=1}^{\infty} \subseteq \mathbb{N}$ and a double sequence $\{b_{ij}\}_{i,j=1}^{\infty} \subseteq l_q$ with pairwise disjoint finite supports and satisfying (3.1). Define the linear operator $\widetilde{T}$ from the span of $\{e_{[i,j],j}\}_{i,j=1}^{\infty}$ to $X \oplus l_q$ by

$$\widetilde{T}(e_{[i,j],j}) = (I - P)T(e_{[i,j],j}) + b_{ij}.$$

Then $\|T - \widetilde{T}(e_{[i,j],j})\| < \epsilon_{i,j}$, for every $1 \le i, j < \omega$. Therefore $\widetilde{T}$ is an into isomorphism and $P\widetilde{T}(e_{[i,j],j}) = b_{ij}$, for every $1 \le i, j < \omega$. □

Proposition 3.2. *Let* X *be a Banach space and* $1 < q < \infty$. *Suppose that* $X \oplus l_q$ *contains a copy of* $l_1(l_q)$. *Then* X *contains a copy of* l_q.

Proof. Let T be an isomorphism from $l_1(l_q)$ into $X \oplus l_q$. Initially observe that for all infinite sequences $N_j \subseteq \mathbb{N}$, $1 \le j < \omega$, $\{e_{i,j}\}_{j=1,\, i\in N_j}^{\infty}$ spans in $l_1(l_q)$ a subspace isometric to $l_1(l_q)$. Thus, thanks to Lemma 3.1 we may suppose that $\{PT(e_{i,j})\}_{i,j=1}^{\infty}$ is a sequence in l_q with pairwise disjoint finite supports.

First of all notice that for any finite set $A \subset \mathbb{N} \times \mathbb{N}$ and sequence $\{a_{n,j}\}_{n,j=1}^{\infty} \subseteq \mathbb{R}$ we have

$$\|\sum_{(n,j)\in A} a_{n,j} PT(e_{n,j})\| \le M(\sum_{(n,j)\in A} |a_{n,j}|^q)^{1/q}, \tag{3.2}$$

where $M = \|P\|\|T\|$.

Now pick $0 < \epsilon < 1$ and $1 \le k < \omega$ satisfying $M\|T^{-1}\|k^{-1/p} < \epsilon$. Observe that for all $\{a_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$ and $1 \le m < \omega$ we have

$$\|\sum_{n=1}^{m} a_n (\frac{1}{k} \sum_{j=1}^{k} e_{n,j})\| = (\sum_{n=1}^{m} |a_n|^q)^{1/q}, \tag{3.3}$$

that is, $\{k^{-1} \sum_{j=1}^{k} e_{n,j}\}_{n=1}^{\infty}$ is equivalent to the l_q basis. Denote by W be the span of these vectors.

Let $\sum_{n=1}^{m} a_n(\frac{1}{k}\sum_{j=1}^{k} T(e_{n,j}))$ be a vector of norm less than or equal to 1. By (3.2) and (3.3) we infer

$$\|P(\sum_{n=1}^{m} a_n(\frac{1}{k}\sum_{j=1}^{k} T(e_{n,j})))\| \leq M(k\sum_{n=1}^{m}\frac{|a_n|^q}{k^q})^{1/q} \leq \frac{M}{k^{1/p}}\|\sum_{n=1}^{m} a_n(\frac{1}{k}\sum_{j=1}^{k} e_{n,j})\|$$

$$\leq \frac{M}{k^{1/p}}\|T^{-1}\|\,\|\sum_{n=1}^{m} a_n(\frac{1}{k}\sum_{j=1}^{k} T(e_{n,j}))\| < \epsilon.$$

Consequently, if I denotes the identity operator of $X \oplus l_q$, then $I - P$ is an isomorphism from a subspace isomorphic to l_q into X. □

Proof of Theorem 1.2. The condition is of course sufficient. Let us show that it is also necessary. To do this, by Theorem 2.4 it is enough to prove that $l_p(\Gamma)$ is an ω_1-quotient of $Y \oplus l_p(\Gamma)$. Since l_p is a uniformly convex space and $(l_p(\Gamma))^n \sim l_p(\Gamma)$ for every $1 \leq n < \omega$, it suffices to prove that

$$C(\xi, Y) \oplus l_p(\Gamma) \not\twoheadrightarrow c_0(l_p),$$

for every $\omega \leq \xi < \omega_1$. But if this is not the case, then by duality and by the separability of $l_1(l_q)$ it follows that $l_1(Y^*) \oplus l_q$ contains a copy of $l_1(l_q)$. Thus, Proposition 3.2 implies that $l_1(Y^*)$ contains a copy of l_q. Then, by a standard gliding hump argument we can prove that Y^* contains a copy of l_q, see for instance [3], a contradiction. This proves the theorem. □

4. On the isomorphic classification of $C(K, Y \oplus l_\infty(\Gamma))$ spaces

In this section we prove Theorem 1.3. First we need to state the following proposition.

Proposition 4.1. *Let A and B be Banach spaces such that there exist a set Λ and $1 < p < \infty$ satisfying*

(a) $B \twoheadrightarrow l_p(\Lambda)$,
(b) $B \not\twoheadrightarrow c_0$,
(c) *for any $\omega \leq \xi < \omega_1$ and bounded linear operator $T : C(\xi, A) \to l_p(\Lambda)$, we have $\mathrm{dens}\, T(C(\xi, A)) < |\Lambda|$.*

Then $l_p(\Lambda)$ is an ω_1-quotient of $X = A \oplus B$.

Proof. Suppose that there exists a bounded linear operator T from $C(\xi, A) \oplus B^n$ onto $c_0(l_p(\Lambda))$ for some $\omega \leq \xi < \omega_1$ and $1 \leq n < \omega$.

Given $1 \leq m < \omega$, we will denote by P_m the natural projection on $c_0(l_p(\Lambda))$ onto the m-th coordinates, that is, $P_m : c_0(l_p(\Lambda)) \to c_0(l_p(\Lambda))$ defined by

$$(x_1, x_2, \ldots, x_m, x_{m+1}, \ldots) \to (0, 0, \ldots, x_m, 0, 0, \ldots).$$

By our hypothesis we deduce that $\mathrm{dens}\, P_m T(C(\xi, A)) < |\Lambda|$, for every $1 \leq m < \omega$. Hence there exists a subset Λ_1 of Λ with $|\Lambda_1| < |\Lambda|$ such that $T(x)(\gamma)(m) = 0$ for every $x \in C(\xi, A)$, $\gamma \notin \Lambda_1$ and $1 \leq m < \omega$. We identify in the natural way $c_0(l_p(\Lambda_1))$ as a subset of $c_0(l_p(\Lambda))$. Let Q be the natural projection from $c_0(l_p(\Lambda))$ onto $c_0(l_p(\Lambda_1))$. So, it is easy to see that the following operator is onto

$$QT_{|B^n} : B^n \to c_0(l_p(\Lambda \setminus \Lambda_1)).$$

Consequently,

$$B^n \twoheadrightarrow c_0.$$

Thus, c_0 is isomorphic to a quotient of B. This contradicts (b) and the proof is complete. □

Proof of Theorem 1.3. Sufficiency is obvious. Let us see necessity. Notice that if Γ is an infinite set, then by [15, Remark 2, p. 203] we have that $l_2(2^{|\Gamma|})$ is isomorphic to a quotient of $l_\infty(\Gamma)$. Moreover, by [6, p. 179] it follows that $l_\infty(\Gamma)$ has no quotient isomorphic to c_0. So, by Proposition 4.1 with $B = l_\infty(\Gamma)$ and $\Lambda = 2^{|\Gamma|}$, we deduce that $l_2(2^{|\Gamma|})$ is an ω_1-quotient of $Y \oplus l_\infty(\Gamma)$. So, by Theorem 2.4 we are done. □

5. On the isomorphic classification of $C(K)$ spaces

In this last section we show that the concept of ω_1-quotient of Banach spaces can also be used to get the isomorphic classifications of certain $C(K)$ spaces for large compact Hausdorff spaces K. Let us start with a closely related result to Theorem 2.4.

Proposition 5.1. *Let X be a Banach space having an ω_1-quotient space Z which is uniformly convex. Write $X = A \oplus B$ as in Definition 2.1. Then for all infinite compact metric spaces K_1 and K_2,*

$$A \oplus C(K_1, B) \sim A \oplus C(K_2, B) \Rightarrow C(K_1) \sim C(K_2).$$

Proof. We consider two cases:

Case 1. K_1 and K_2 are countable. Pick ξ and η infinite countable ordinals such that $C(K_1)$ is isomorphic to $C(\xi)$ and $C(K_2)$ is isomorphic to $C(\eta)$. Without loss of generality we may assume that $\xi \leq \eta$. Then,

$$A \oplus C(\xi, B) \sim A \oplus C(\eta, B) \twoheadrightarrow C(\eta, B) \twoheadrightarrow C(\eta, Z).$$

Hence by Proposition 2.5 and (1.2) we infer that $C(K_1)$ is isomorphic to $C(K_2)$.

Case 2. K_2 is uncountable. We will show that $C(K_1)$ is isomorphic to $C(K_2)$ by proving that K_1 is uncountable. Otherwise, there exists a countable ordinal ξ such that $C(K_1)$ is isomorphic to $C(\xi)$. Thus,

$$A \oplus C(\xi, B) \sim A \oplus C(K_2, B) \twoheadrightarrow C(\xi^\omega, B) \twoheadrightarrow C(\xi^\omega, Z)$$

a contradiction by Proposition 2.5 and the proof of proposition is complete. □

Recall that a topological space S is said to be dispersed if every nonempty subset of S contains a relatively isolated point. Furthermore, the topological weight of a topological space K is the smallest cardinal $\mathfrak{m}$ such that there exists a base of open subsets of K of cardinality $\mathfrak{m}$.

Theorem 5.2. *Let Γ be an infinite set and S a dispersed compact Hausdorff space or an infinite compact Hausdorff space having topological weight strictly less than $2^{|\Gamma|}$. Then for any infinite compact metric spaces K_1 and K_2,*

(a) $C(K_1 \times (S \oplus \beta\Gamma)) \sim C(K_2 \times (S \oplus \beta\Gamma)) \Leftrightarrow C(K_1) \sim C(K_2)$.
(b) $C(S \oplus (K_1 \times \beta\Gamma)) \sim C(S \oplus (K_2 \times \beta\Gamma)) \Leftrightarrow C(K_1) \sim C(K_2)$.

Proof. Of course, the condition $C(K_1) \sim C(K_2)$ is sufficient for both statements of the proposition. We will show that this condition is also necessary. First of all observe that

$$C(K \times (S \oplus \beta\Gamma)) \sim C(K, C(S) \oplus l_\infty(\Gamma)),$$

and

$$C(S \oplus (K \times \beta\Gamma)) \sim C(S) \oplus C(K, l_\infty(\Gamma)),$$

for every compact Hausdorff space K.

Set $A = C(S)$, $B = l_\infty(\Gamma)$ and $Z = l_2(2^{|\Gamma|})$. In view of Theorem 2.4 and Proposition 5.1, it suffices to show that $l_2(2^{|\Gamma|})$ is an ω_1-quotient of $X = C(S) \oplus l_\infty(\Gamma)$. We distinguish two cases:

Case 1. S is dispersed. We know that $l_2(2^{|\Gamma|})$ is isomorphic to a quotient of $l_\infty(\Gamma)$. On the other hand, notice that for any ordinal $\omega \leq \xi < \omega_1$ the compact space $[0, \xi] \times S$ is also dispersed. Moreover, it is well-known that any bounded liner operator T from $C([0, \xi] \times S)$ to $l_2(2^{|\Gamma|})$ is compact [6, Theorem 15, p. 159] and [7, p. 647]. Therefore, by [18, Proposition 7.6.5] $dens\, T(C([0, \xi] \times S)) \leq \aleph_0 < 2^{|\Gamma|}$. Thus, it is enough to apply Proposition 4.1 with $\Lambda = 2^{|\Gamma|}$.

Case 2. The topological weight of S is strictly less than $2^{|\Gamma|}$. In this case, $dens\, C(S) < 2^{|\Gamma|}$ [18, Proposition 7.6.3] and by Proposition 4.1 with $\Lambda = 2^{|\Gamma|}$ we are done. □

Recall that a compact Hausdorff space Ω is called Stonean (or extremally disconnected) if the closure of every open set is open, see [6, Definition 7, p. 154].

Theorem 5.3. *Let Ω be an infinite Stonean space and S a dispersed compact Hausdorff space or an infinite compact Hausdorff space having topological weight strictly less than $2^{\aleph_0}$. Then for any infinite compact metric spaces K_1 and K_2,*

(a) $C(K_1 \times (S \oplus \Omega)) \sim C(K_2 \times (S \oplus \Omega)) \Leftrightarrow C(K_1) \sim C(K_2)$.
(b) $C(S \oplus (K_1 \times \Omega)) \sim C(S \oplus (K_2 \times \Omega)) \Leftrightarrow C(K_1) \sim C(K_2)$.

Proof. Let us show the non-trivial implications. By [6, p. 156] $C(\Omega)$ has a quotient isomorphic to l_∞. Moreover, l_∞ has a quotient isomorphic to $l_2(2^{\aleph_0})$. So, it is enough to proceed as in the proof of Theorem 5.2. □

Remark 5.4. Regarding the statements of Theorem 2.4 and Proposition 5.1 observe that if $X = A \oplus B$ as in Definition 2.1, then we do not have necessarily

$$C(K, X) \sim A \oplus C(K, B),$$

for every infinite compact metric space K.

Indeed, on the one hand by Proposition 3.2 and [1, Theorem 6.4.19i] we deduce that the l_p space with $1 < p < 2$ is an ω_1-quotient of $X = l_\infty \oplus l_p$.

On the other hand, since $l_2(2^{\aleph_0})$ is a quotient of l_∞, we conclude by Proposition 4.1 that $l_2(2^{\aleph_0})$ is an ω_1-quotient of $l_p \oplus l_\infty$. So, by the item (c) of Definition 2.1 we infer

$$C(\omega, l_p) \oplus l_\infty \not\twoheadrightarrow C(\omega, l_2(2^{\aleph_0})). \tag{5.1}$$

Consequently, we cannot have

$$C(\omega, l_\infty \oplus l_p) \sim l_\infty \oplus C(\omega, l_p),$$

otherwise,

$$C(\omega, l_p) \oplus l_\infty \sim C(\omega, l_\infty \oplus l_p) \twoheadrightarrow C(\omega, l_\infty) \twoheadrightarrow C(\omega, l_2(2^{\aleph_0})),$$

a contradiction by (5.1).

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