London-van der Waals interaction in different geometries

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Abstract. We consider the attractive potential energy between two particles in the form of London-van der Waals interaction, $u = -\lambda/r^6$. The constant λ has dimensions of energy $\times$ (distance)^6. The interaction between systems is evaluated in different symmetries. In particular, the Hamaker potential between spheres is obtained.

1 Introduction

We consider here the attractive potential between two point particles,

$$u = -\frac{\lambda}{r^6}\,, \tag{1}$$

known as London-van der Waals attraction [3, 4, 5, 18]. The interaction energy between systems in different geometries is evaluated. In particular we obtain the Hamaker potential energy between two spherical shells.

1- Particle - line

We consider here a particle at a distance a from a straight line, located on the z axis (fig. 1). The energy between the particle and a line element dz of the straight line is,

$$du = -\frac{\lambda}{r^6}\rho_l dz = -\frac{\lambda}{(z^2+a^2)^3}\rho_l dz\,,$$

where ρ_l is the linear density of particles on the line. The number density ρ_l is considered uniform. The energy between the particle and the line is,

$$u = \int du = -\lambda\rho_l \int_{-L_1}^{+L_2} \frac{dz}{(z^2+a^2)^3}\,.$$

Integrating the above expression we have,

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$$u = -\lambda\rho_l \left[\frac{3z^3 + 5za^2}{8a^4(z^2+a^2)^2} + \frac{3}{8a^5}\tan^{-1}\frac{z}{a} \right]_{-L_1}^{+L_2} ,$$

or,

$$\begin{aligned} u &= -\lambda\rho_l \left[\frac{3L_2^3 + 5L_2 a^2}{8a^4(L_2^2+a^2)^2} - \frac{-3L_1^3 - 5L_1 a^2}{8a^4(L_1^2+a^2)^2} \right. \\ & \left. + \frac{3}{8a^5}\tan^{-1}\frac{L_2}{a} - \frac{3}{8a^5}\tan^{-1}\frac{-L_1}{a} \right]_{-L_1}^{+L_2} . \end{aligned} \quad (2)$$

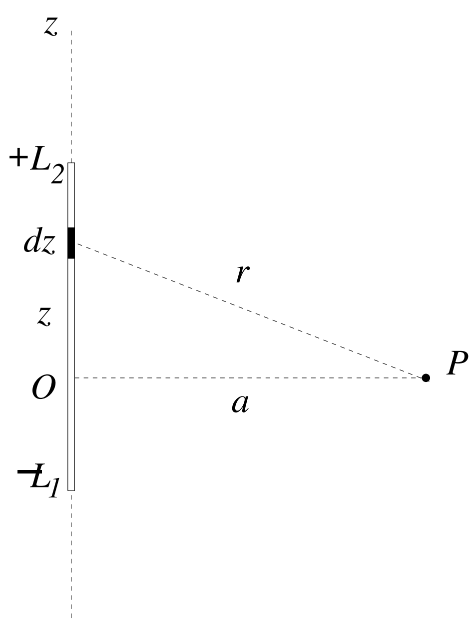

Fig. 1. A particle interacting with a line on the z axis.

For an infinite linte we take the limit $L_1, L_2 \to \infty$,

$$u = -\frac{3\pi\lambda\rho_l}{8a^5} . \quad (3)$$

2- Particle - plane

We consider now a particle on the z axis, interacting with a plane located in the xy plane (fig. 2). The interaction energy between the particle and a surface element da of the plane is,

$$du = -\frac{\lambda}{r^6}\rho_s da ,$$

with $r^2 = x^2 + y^2 + z^2$ and ρ_s is the surface density of particles on the plane, consideraded uniform. Integrating the above expression we have the energy,

$$u = -\lambda \rho_s \int \frac{dx\, dy}{(x^2 + y^2 + z^2)^3} \,.$$

Integrating in x first,

$$\begin{aligned} u &= -\lambda \rho_s \int_{-b}^{+b} dy \int_{-a}^{+a} \frac{dx}{(x^2 + y^2 + z^2)^3} \,, \\ u &= -2\lambda \rho_s \int_{-b}^{+b} dy \left[\frac{3a^3 + 5a(y^2 + z^2)}{8(y^2 + z^2)^2 (a^2 + y^2 + z^2)^2} \right. \\ &\left. + \frac{3}{8(y^2 + z^2)^{5/2}} \tan^{-1} \frac{a}{(y^2 + z^2)^{1/2}} \right] \,. \end{aligned} \tag{4}$$

How to evaluate the above integral? First we consider the limit of an infinite plane. Taking the limit $a, b \to \infty$ we have,

$$u = -\lambda \rho_s \frac{3\pi}{4} \int_0^{\infty} \frac{dy}{(y^2 + z^2)^{5/2}} \,.$$

Evaluating the above integral we obtain,

$$u = -\frac{\pi \lambda \rho_s}{2 z^4} \,. \tag{5}$$

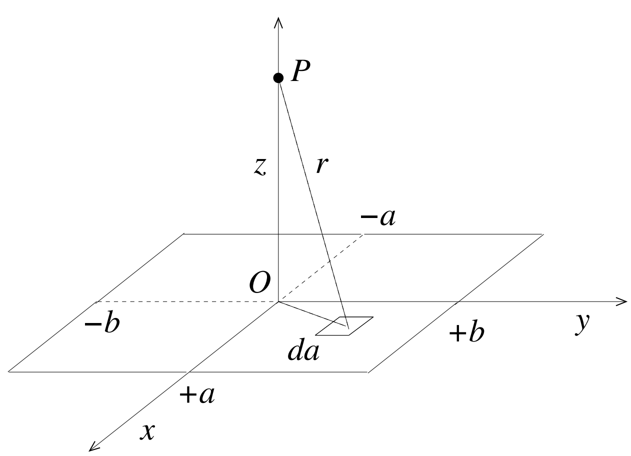

Fig. 2. A particle interacting with a plane located in the xy plane.

3- Particle - circle

We consider now a particle in the z axis and a circle of radius a in the plane xy with center at the origin (fig. 3). The interaction energy between the particle and a line element ds of the circle is,

$$du = -\frac{\lambda}{r^6}\rho_l ds\,.$$

The linear density of particles ρ_l is taken to be uniform. Substituting $ds = ad\varphi$, where φ is the usual angle in polar coordinates, we have,

$$du = -\frac{\lambda}{r^6}\rho_l a d\varphi\,.$$

Integrating the above expression on the circle,

$$u = \int du = -\frac{\lambda}{r^6}\rho_l a \int_0^{2\pi} d\varphi = -\frac{\lambda}{r^6}2\pi\rho_l a\,,$$

since r is the same for all points on the circle. Substituting $r^2 = z^2 + r^2$,

$$u = -\frac{2\pi\rho_l a\lambda}{(z^2+a^2)^3}\,. \tag{6}$$

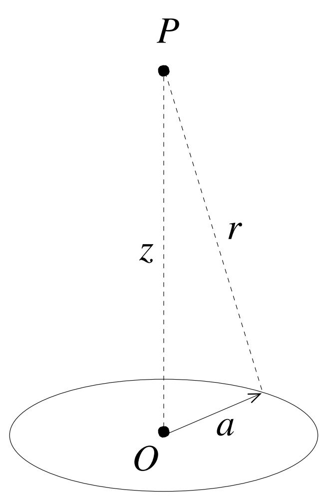

Fig. 3. A particle interacting with a circle of radius a.

4- Particle - disc

We consider now a particle in the z axis interacting with a disc in the xy plan, with the center at the origin (fig. 4). Using equation (6),

$$du = -\frac{2\pi\rho_l x\lambda}{(z^2+x^2)^3}\,, \tag{7}$$

where ρ_l is linear density of particles on the circular strip of infinitesimal thickness dx,

$$\rho_l = \frac{N}{2\pi x}\,.$$

The number N of particles on the infinitesimal strip is,

$$N = \rho_s 2\pi x dx\,,$$

where ρ_s is the surface density of particles on the disc, taken to be uniform. Hence,

$$\rho_l = \frac{N}{2\pi x} = \frac{\rho_s 2\pi x dx}{2\pi x} = \rho_s dx\,,$$

and the equation (7) becomes,

$$du = -\frac{2\pi\lambda\rho_s x dx}{(z^2+x^2)^3}\,. \tag{8}$$

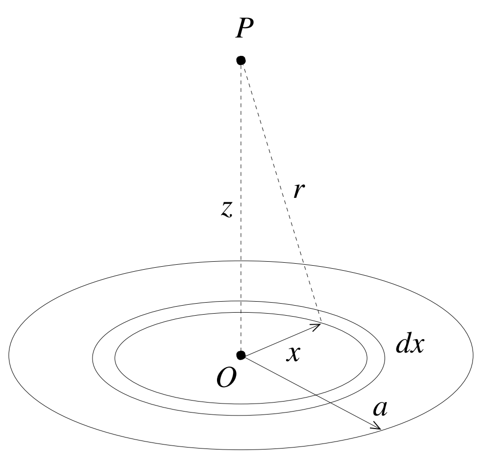

Fig. 4. A particle interacting with a disc of radius a.

The total energy is obtained integrating the above expression,

$$u = \int du = \frac{1}{2}\pi\lambda\rho_s\left[\frac{1}{(z^2+a^2)^2} - \frac{1}{z^4}\right]\,,$$

or,

$$u = -\frac{1}{2}\pi\lambda\rho_s \frac{a^2(2z^2 + a^2)}{z^4(z^2 + a^2)^2}\,. \qquad (9)$$

5- Particle - infinitesimal shell

We consider here a particle on the z axis interacting with a spherical shell of infinitesimal thickness and radius a, with center at the origin (fig. 5). The interaction energy between the particle and dA is,

$$du = -\frac{\lambda}{r^6}\rho_s dA\,.$$

The surface element is $dA = 2\pi a^2 \mathrm{sen}\,\theta\, d\,\theta$ and we have also,

$$r^2 = a^2 + z^2 - 2az\cos\theta\,,$$

hence,

$$u = \int du = -2\pi a^2\lambda\rho_s \int_0^\pi \frac{\mathrm{sen}\,\theta\, d\,\theta}{(a^2 + z^2 - 2az\cos\theta)^3}\,.$$

We evaluate the above integral changing variables, $x = \cos\theta$, $dx = -\mathrm{sen}\,\theta\, d\,\theta$. We obtain,

$$u = -4\pi\lambda\rho_s \frac{a^2(z^2 + a^2)}{(z^2 - a^2)^4}\,. \qquad (10)$$

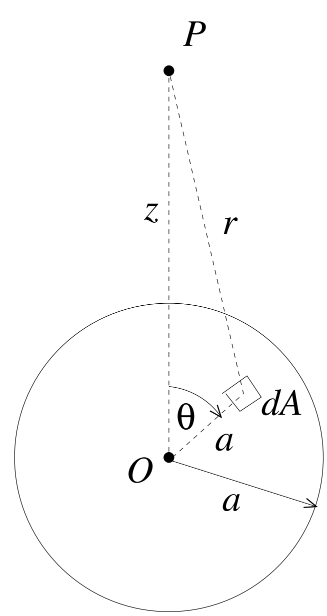

Fig. 5. A particle interacting with a shell of radius a.

6- Particle - sphere (as a set of discs)

We consider a particle on the z axis, interacting with a sphere of radius a, with center at the origin (fig. 6), represented as a set of discs. Using the expression before (9),

$$du = \frac{1}{2}\pi\lambda\rho_s \left[\frac{1}{[(z-z')^2 + R^2]^2} - \frac{1}{(z-z')^4} \right] ,$$

where ρ_s is the surface density of particles on the disc. If ρ_v is the volume density of particles on the sphere,

$$\rho_s = \frac{N}{\pi R^2} = \frac{\pi R^2 \rho_v dz'}{\pi R^2} = \rho_v dz' .$$

Therefore,

$$u = \int du = \frac{1}{2}\pi\lambda\rho_v \int_{-a}^{+a} dz' \left[\frac{1}{[(z-z')^2 + R^2]^2} - \frac{1}{(z-z')^4} \right] .$$

Evaluating the above integral we obtain,

$$u = -\frac{4\pi\lambda\rho_v a^3}{3(z^2 - a^2)^3} . \tag{11}$$

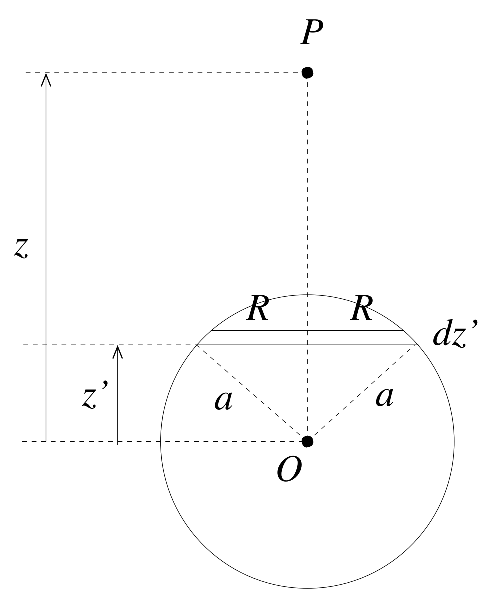

Fig. 6. A particle interacts with a sphere of radius a, considered as a set of discs.

7- Particle - sphere (as a set of shells)

We consider the interaction between a particle with a sphere, this time considered as a set of shells (fig. 7). Using (10),

$$du = -4\pi\lambda\rho_s \frac{r^2(z^2+r^2)}{(z^2-r^2)^4}\,,$$

where ρ_s is the surface number density of particles in a shell. In terms of ρ_v,

$$\rho_s = \frac{N}{4\pi r^2} = \frac{\rho_v 4\pi r^2 dr}{4\pi r^2} = \rho_v dr\,.$$

Therefore,

$$du = -4\pi\lambda\rho_v dr \frac{r^2(z^2+r^2)}{(z^2-r^2)^4}\,.$$

Integrating the above expression we have the total energy,

$$u = \int du = -4\pi\lambda\rho_v \int_0^a dr \frac{r^2(z^2+r^2)}{(z^2-r^2)^4}\,.$$

The result is,

$$u = -\frac{4\pi\lambda\rho_v a^3}{3(z^2-a^2)^3}\,, \tag{12}$$

as the equation (11), as expected.

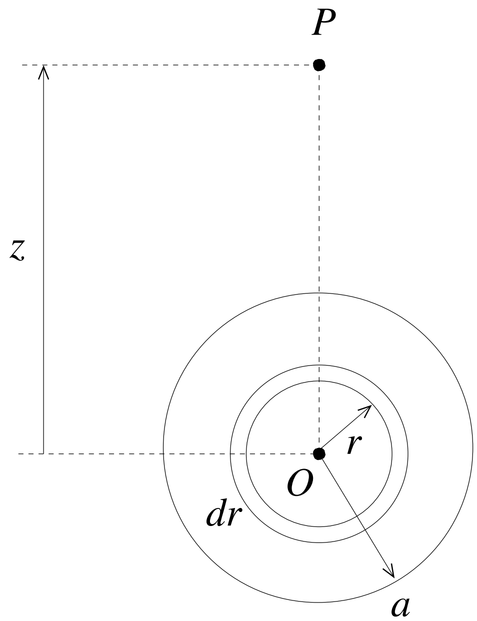

Fig. 7. A particle interacts with a sphere of radius a, considered as a set of infinitesimal shells.

8- Particle - finite shells

We consider now the interaction of a particle with a spherical shell of finite thickness, with internal radius a and external radius b (fig. 8). Using (11) or (12), we can write the interaction energy as,

$$u = u(b) - u(a)\,,$$

in which we subtract the interaction energy of a sphere with radius a with the particle from the interaction energy of a sphere with radius b with the particle. Therefore,

$$u = -\frac{4\pi\lambda\rho_v}{3}\left[\frac{b^3}{(z^2-b^2)^3} - \frac{a^3}{(z^2-a^2)^3}\right]\,. \tag{13}$$

It is of interest to consider the limit of a shell with infinitesimal thickness. We do $b = a + \delta$, and then take the limit $\delta \to 0$. The product $\rho_v\delta$ is the surface density ρ_s. After some algebra we obtain the expression (10).

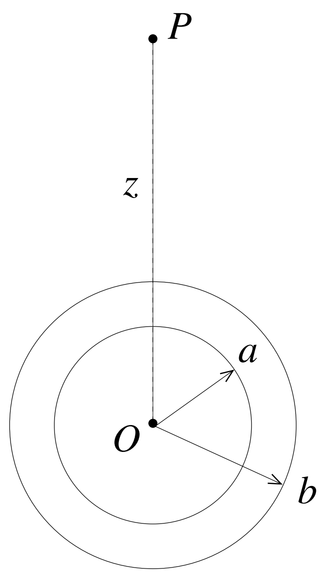

Fig. 8. A particle interacts with a shell of finite thickness, with internal radius a and external radius b.

9- Circle - infinitesimal shell

We consider now the interaction between a circle of radius b and an infinitesimal shell of radius a (fig. 9). All the points of the circle are at the same distance r from the shell, with r given by,

$$r^2 = z^2 + b^2 \, .$$

We can use the expression (10),

$$du = -4\pi\lambda\rho_s\rho_l ds \frac{a^2(r^2+a^2)}{(r^2-a^2)^4} \, ,$$

where ρ_s is the surface density of particles over the shell and ρ_l is the linear density over the circle, both uniform, and ds is the line element on the circle. Substituting $ds = bd\varphi$ and the expression for r^2, we obtain,

$$du = -4\pi\lambda\rho_s\rho_l b d\varphi \frac{a^2(z^2+b^2+a^2)}{(z^2+b^2-a^2)^4} \, ,$$

and hence,

$$u = -8\pi^2\lambda\rho_s\rho_l b \frac{a^2(z^2+b^2+a^2)}{(z^2+b^2-a^2)^4} \, . \tag{14}$$

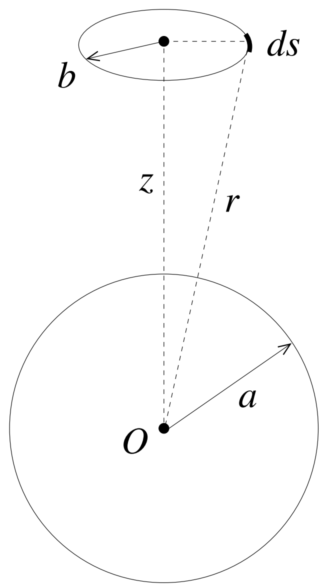

Fig. 9. A circle of radius b interacts with a shell of infinitesimal thickness of radius a.

10- Circle - sphere

We consider now the interaction between a circle of radius b and a sphere of radius a. The situation is the same of figure 9. All points of the circle are at the same distance r from the sphere, with r given by,

$$r^2 = z^2 + b^2 \,.$$

We can use (11) or (12),

$$du = -\frac{4\pi\lambda\rho_v a^3}{3(r^2 - a^2)^3}\rho_l ds \,,$$

where ρ_v is the volume density of particles over the sphere and ρ_l is the linear density of particles over the circle, both uniform, and ds is the line element of the circle. Substituting $ds = bd\varphi$ and the expression for r^2, we obtain,

$$du = -\frac{4\pi\lambda\rho_v a^3}{3(z^2 + b^2 - a^2)^3}\rho_l bd\varphi \,,$$

and hence,

$$u = -\frac{8\pi^2\lambda\rho_v\rho_l ba^3}{3(z^2 + b^2 - a^2)^3} \,. \qquad (15)$$

11- Circle - finite shell

We consider a circle of radius c interacting with a finite shell of internal radius a and external radius b (fig. 10). We have,

$$u = u(b) - u(a)\,,$$

where $u(b)$ is the energy between a circle and a sphere of radius b, and similarly for $u(a)$. Using the expression (15),

$$u = -\frac{8\pi^2\lambda\rho_v\rho_l c}{3}\left[\frac{b^3}{(z^2+c^2-b^2)^3} - \frac{a^3}{(z^2+c^2-a^2)^3}\right]. \tag{16}$$

It is of interest to consider the limit of an infinitesimal shell, as previously in the case of a particle. We do $b = a + \delta$, and after we take the limit $\delta \to 0$. The product $\rho_v\delta$ is the surface density of particles ρ_s. After some algebra we obtain,

$$u = -8\pi^2\lambda\rho_s\rho_l \frac{ca^2(z^2+c^2+a^2)}{(z^2+c^2-a^2)^4}\,,$$

that is equivalent to the expression (14).

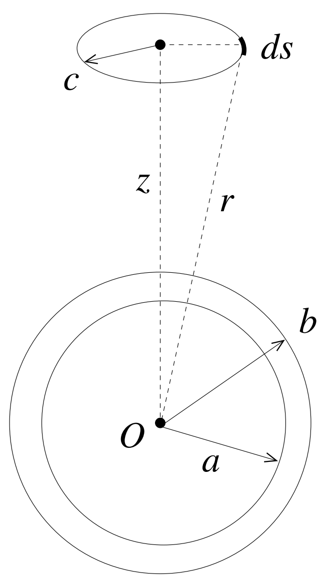

Fig. 10. A circle of radius c interacts with a spherical shell with internal radius a and external radius b.

12- Infinitesimal shell - infinitesimal shell

We consider now the interaction between two infinitesimal shells (fig. 11), representing one of them as a set of circles. Using the expression (14), we write the interaction between a shell of radius a and a circle of radius r over the shell radius b, in the form,

$$du = -8\pi^2 \lambda \rho_s^a \rho_s^b dz' \frac{ra^2(z'^2 + r^2 + a^2)}{(z'^2 + r^2 - a^2)^4} \,,$$

where we used the linear density over the circle as $\rho_l = \rho_s^b dz'$. We see in the figure that,

$$r^2 + (z - z')^2 = b^2 \,,$$

hence,

$$du = -8\pi^2 \lambda \rho_s^a \rho_s^b dz' \frac{\sqrt{b^2 - (z - z')^2} a^2 (b^2 - z^2 + 2zz' + a^2)}{(b^2 - z^2 + 2zz' - a^2)^4} \,.$$

How to do the above integral?

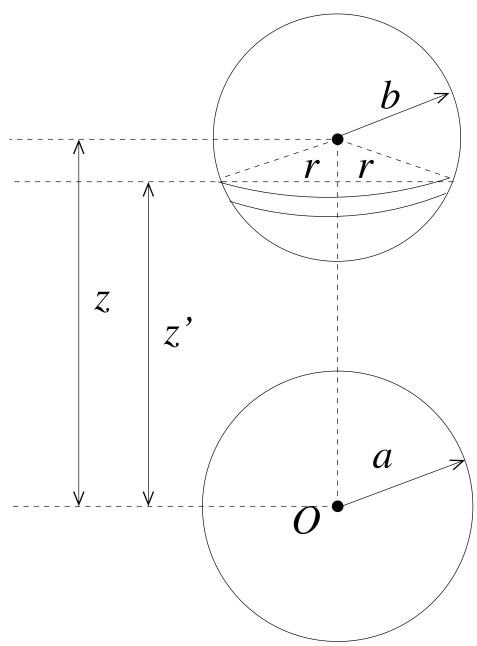

Fig. 11. An infinitesimal shell of radius a interacts with other infinitesimal shell of radius b.

We evaluate u in another way. Figure 12 shows a surface element dA over one of the shells. Using expression (10) we write the energy interaction between a shell of radius b and the element dA as,

$$du = -4\pi\lambda\rho_s^b \frac{b^2(r^2+b^2)}{(r^2-b^2)^4}\rho_s^a dA\,.$$

Substituting $dA = a^2 \mathrm{sen}\,\theta d\,\theta d\,\varphi = 2\pi a^2 \mathrm{sen}\,\theta d\,\theta$, and,

$$r^2 = z^2 + a^2 - 2az\cos\theta\,,$$

we obtain,

$$du = -8\pi^2\lambda a^2 b^2 \rho_s^a \rho_s^b \mathrm{sen}\,\theta d\,\theta \frac{(z^2+a^2+b^2-2az\cos\theta)}{(z^2+a^2-b^2-2az\cos\theta)^4}\,.$$

Integrating the above expression we have the total energy u. Changing variables $x = \cos\theta$ we have,

$$u = -8\pi^2\lambda a^2 b^2 \rho_s^a \rho_s^b \int_{-1}^{+1} dx \frac{z^2+a^2+b^2-2azx}{(z^2+a^2-b^2-2azx)^4}\,.$$

The result is,

$$u = -\frac{2\pi^2}{3z}\lambda\rho_s^a\rho_s^b a b^2\,\mathcal{I}\,, \tag{17}$$

with,

$$\mathcal{I} = \frac{3(z-a)^2+b^2}{[(z-a)^2-b^2]^3} - \frac{3(z+a)^2+b^2}{[(z+a)^2-b^2]^3}\,. \tag{18}$$

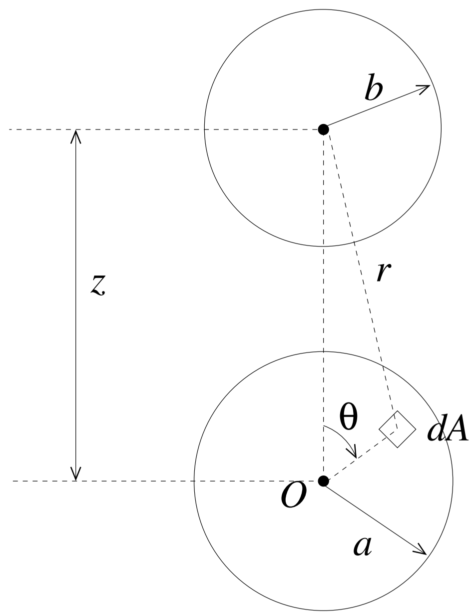

Fig. 12. An infinitesimal shell of radius a interacts with another infinitesimal shell of radius b. A surface element dA over one of the shells is shown.

13- Infinitesimal shell - sphere

Here we evaluate the interaction between an infinitesimal shell of radius a and a sphere of radius b (fig. 13). We consider the sphere as a set of shells. Using equations (17) and (18), we write the interaction energy between the shell of radius a and a shell of infinitesimal thickness dr and radius r as,

$$du = -8\pi^2\lambda a^2 r^2 \rho_s^a \rho_v^b dr\, \mathcal{I}\,,$$

with,

$$\mathcal{I} = \frac{3(z-a)^2 + r^2}{12az[(z-a)^2 - r^2]^3} - \frac{3(z+a)^2 + r^2}{12az[(z+a)^2 - r^2]^3}\,.$$

Using $\rho_s = \rho_v^b dr$ as the surface density over the infinitesimal shell of radius r and thickness dr. We need to integrate the above expression in dr to obtain u,

$$u = -\frac{8\pi^2\lambda a^2}{12az}\rho_s^a\rho_v^b(\mathcal{I}_1 - \mathcal{I}_2)\,,$$

with

$$\mathcal{I}_1 = \int_0^b drr^2 \frac{3(z-a)^2 + r^2}{[(z-a)^2 - r^2]^3} \,,$$
$$\mathcal{I}_2 = \int_0^b drr^2 \frac{3(z+a)^2 + r^2}{[(z+a)^2 - r^2]^3} \,.$$

Evaluating $\mathcal{I}_1$ and $\mathcal{I}_2$ we obtain,

$$\mathcal{I}_1 = \frac{b^3}{[(z-a)^2 - b^2]^2} \,,$$
$$\mathcal{I}_2 = \frac{b^3}{[(z+a)^2 - b^2]^2} \,.$$

The final expression for the interaction energy between an infinitesimal shell of radius a and a sphere of radius b is then,

$$u = -\frac{2\pi^2 \lambda a b^3}{3z} \rho_s^a \rho_v^b \left\{ \frac{1}{[(z-a)^2 - b^2]^2} - \frac{1}{[(z+a)^2 - b^2]^2} \right\} \,. \tag{19}$$

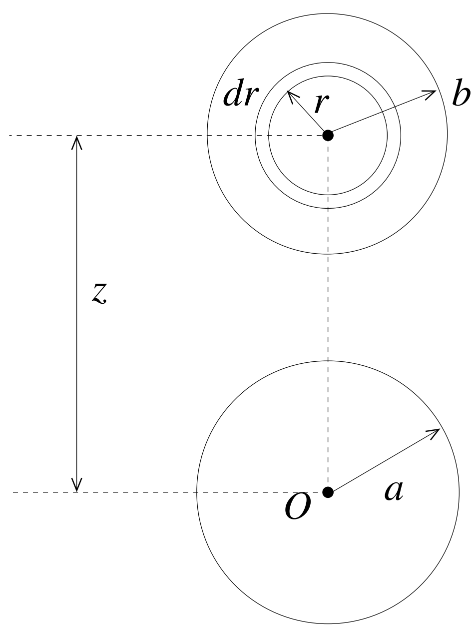

Fig. 13. An infinitesimal shell of radius a interacts with a sphere of radius b. We consider the sphere as a set of infinitesimal shells.

14- Sphere - sphere

We evaluate now the interaction between two spheres, considering one of them as a set of infinitesimal shells (fig. 14). This figure is identical to figure 13, but here we have two spheres, while in figure 13 we have a shel and a sphere. Using the expression (19) we write,

$$du = -\frac{2\pi^2\lambda a^3}{3z}\rho_v^a\rho_v^b r\,dr\left\{\frac{1}{[(z-r)^2-a^2]^2} - \frac{1}{[(z+r)^2-a^2]^2}\right\}.$$

The total energy is then given by,

$$u = -\frac{2\pi^2\lambda a^3}{3z}\rho_v^a\rho_v^b(\mathcal{I}_1 - \mathcal{I}_2)\,,$$

with

$$\mathcal{I}_1 = \int_0^b \frac{r\,dr}{[(z-r)^2-a^2]^2}\,,$$
$$\mathcal{I}_2 = \int_0^b \frac{r\,dr}{[(z+r)^2-a^2]^2}\,.$$

Evaluating the above integrals,

$$\mathcal{I}_1 = -\frac{1}{2a^2} + \frac{1}{2[(z-b)^2-a^2]}\left[\frac{z(z-b)}{a^2} - 1\right]$$
$$+\frac{z}{4a^3}\ln\frac{(z-b-a)(z+a)}{(z-b+a)(z-a)}\,,$$
$$\mathcal{I}_2 = -\frac{1}{2a^2} + \frac{1}{2[(z+b)^2-a^2]}\left[\frac{z(z+b)}{a^2} - 1\right]$$
$$+\frac{z}{4a^3}\ln\frac{(z+b-a)(z+a)}{(z+b+a)(z-a)}\,.$$

The result for u is hence,

$$\begin{aligned} u &= -\frac{\pi^2\lambda a^3}{3z}\rho_v^a\rho_v^b\left\{\frac{1}{(z-b)^2-a^2}\left[\frac{z(z-b)}{a^2} - 1\right]\right. \\ &\quad -\frac{1}{(z+b)^2-a^2}\left[\frac{z(z+b)}{a^2} - 1\right] \\ &\quad \left.+\frac{z}{2a^3}\ln\frac{(z-b-a)(z+b+a)}{(z-b+a)(z+b-a)}\right\}, \end{aligned}$$

or, in a symmetric form in a, b,

$$\begin{aligned} u &= -\frac{\pi^2}{3}\lambda\rho_v^a\rho_v^b\,ab \\ &\times\left\{\frac{1}{z^2-(a-b)^2}+\frac{1}{z^2-(a+b)^2}\right. \\ &\left.+\frac{1}{2ab}\ln\frac{z^2-(b+a)^2}{z^2-(b-a)^2}\right\}\,. \end{aligned} \tag{20}$$

This expression is identical to that obtained by Hamaker [46].

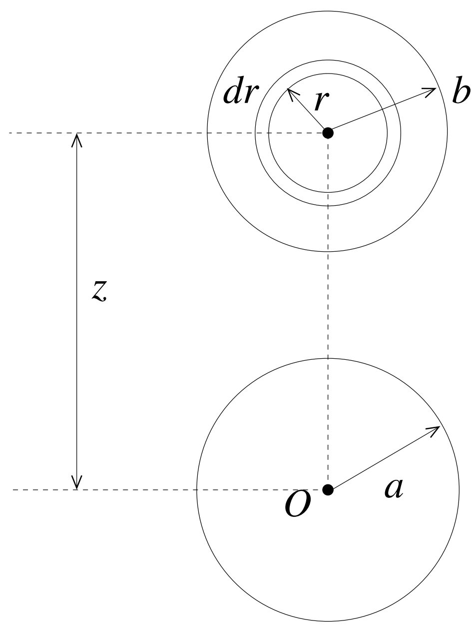

Fig. 14. A sphere of radius a interacts with a sphere of radius b. We consider the sphere of radius b as a set of infinitesimal shells.

15- Infinitesimal shell - finite shell

We consider here an infinitesimal shell of radius c interacting with a finite shell of internal radius a and external radius b. The interaction energy may be writen as the energy with a sphere of radius raio b minus the energy with a sphere of radius a. Using the expression (19),

$$
\begin{aligned}
u &= u(b) - u(a)\,, \\
&= -\frac{2\pi^2\lambda c b^3}{3z}\rho_s^c\rho_v \left\{ \frac{1}{[(z-c)^2-b^2]^2} - \frac{1}{[(z+c)^2-b^2]^2} \right\} \\
&\quad + \frac{2\pi^2\lambda c a^3}{3z}\rho_s^c\rho_v \left\{ \frac{1}{[(z-c)^2-a^2]^2} - \frac{1}{[(z+c)^2-a^2]^2} \right\}\,. \qquad (21)
\end{aligned}
$$

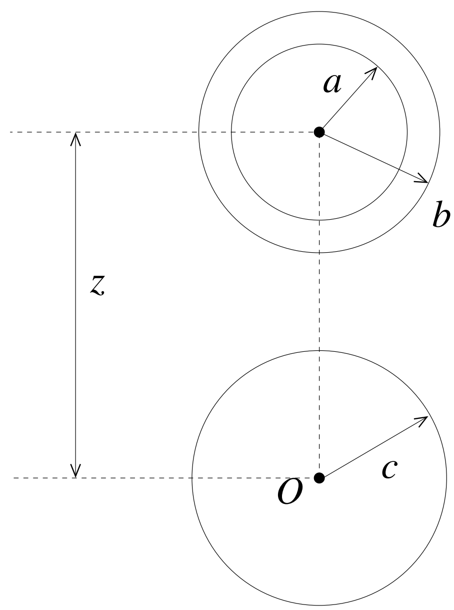

Fig. 15. An infinitesimal shell of radius c interacts with a finite shell of internal radius a and external radius b.

16- Finite shell - sphere

We evaluate now the interaction energy between a sphere of radius c and a finite shell of internal radius a and external radius b. The situation is shown in fig. 16, which is identical to figure 15, but now we have a sphere instead of an infinitesimal shell. Using equation (20), we write this energy as the difference betwenn the energies of a sphere of radius b and a sphere of radius a, with a sphere of radius c,

$$
\begin{aligned}
u &= u(b) - u(a)\,, \\
&= -\frac{\pi^2}{3}\lambda\rho_v^b\rho_v^c\, bc\left\{\frac{1}{z^2-(b-c)^2} + \frac{1}{z^2-(b+c)^2}\right. \\
&\quad \left. +\frac{1}{2bc}\ln\frac{z^2-(b+c)^2}{z^2-(b-c)^2}\right\} \\
&\quad +\frac{\pi^2}{3}\lambda\rho_v^a\rho_v^c\, ac\left\{\frac{1}{z^2-(a-c)^2} + \frac{1}{z^2-(a+c)^2}\right. \\
&\quad \left. +\frac{1}{2ac}\ln\frac{z^2-(a+c)^2}{z^2-(a-c)^2}\right\}\,. \qquad (22)
\end{aligned}
$$

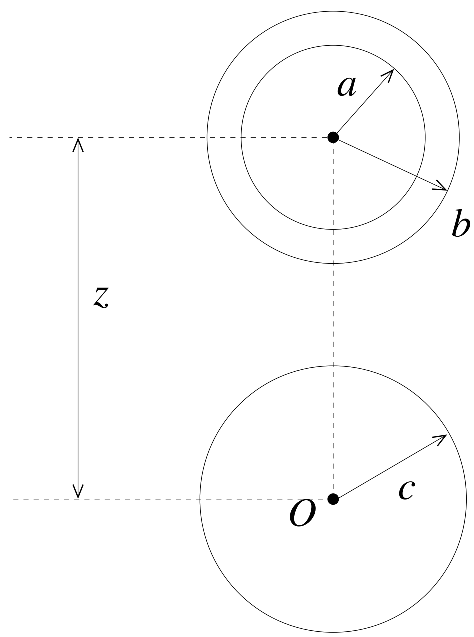

Fig. 16. A sphere of radius c interacts with a finite shell of internal radius a and external radius b.

17- Finite shell - Finite shell

We here evaluate the energy interaction between finite shells (fig. 17). Denoting by u_{ij} the energy between two spheres of radius i and j, we may write the energy between the shells as,

$$
u = u_{bd} - u_{bc} + u_{ac} - u_{ad}\,.
$$

Using equation (20),

$$
\begin{aligned}
u \;=\; & -\frac{\pi^2}{3}\lambda\rho_v^b\rho_v^d\, bd\left\{\frac{1}{z^2-(b-d)^2}+\frac{1}{z^2-(b+d)^2}\right. \\
& \left.+\frac{1}{2bd}\ln\frac{z^2-(b+d)^2}{z^2-(b-d)^2}\right\} \\
& +\frac{\pi^2}{3}\lambda\rho_v^b\rho_v^c\, bc\left\{\frac{1}{z^2-(b-c)^2}+\frac{1}{z^2-(b+c)^2}\right. \\
& \left.+\frac{1}{2bc}\ln\frac{z^2-(b+c)^2}{z^2-(b-c)^2}\right\} \\
& -\frac{\pi^2}{3}\lambda\rho_v^a\rho_v^c\, ac\left\{\frac{1}{z^2-(a-c)^2}+\frac{1}{z^2-(a+c)^2}\right. \\
& \left.+\frac{1}{2ac}\ln\frac{z^2-(a+c)^2}{z^2-(a-c)^2}\right\} \\
& +\frac{\pi^2}{3}\lambda\rho_v^a\rho_v^d\, ad\left\{\frac{1}{z^2-(a-d)^2}+\frac{1}{z^2-(a+d)^2}\right. \\
& \left.+\frac{1}{2ad}\ln\frac{z^2-(a+d)^2}{z^2-(a-d)^2}\right\}\;. \qquad (23)
\end{aligned}
$$

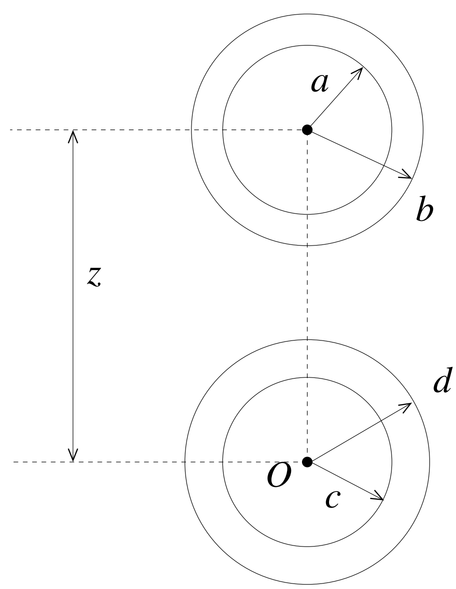

Fig. 17. A finite shell of internal radius a and external radius b interacts with other finite shell, of internal radius c and external radius d.

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