9 - A equação da onda em coordenadas cartesianas

A equação de onda é,

$$\frac{\partial^2 u}{\partial t^2} = v^2 \nabla^2 u \,. \tag{1}$$

Em coordenadas cartesianas,

$$\frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \,. \tag{2}$$

Substituindo,

$$u(\mathbf{r}, t) = F(\mathbf{r})T(t) \;, \tag{3}$$

temos,

$$\frac{1}{v^2} F T'' = T \nabla^2 F \;,$$

ou,

$$\frac{1}{v^2}\frac{T''}{T} = \frac{\nabla^2 F}{F} = -k^2 \,.$$

Obtemos assim as equações,

$$\frac{d^2 T}{dt^2} + (kv)^2 T = 0 \,, \tag{4}$$

e,

$$\nabla^2 F + k^2 F = 0 \,. \tag{5}$$

Essa equação é idêntica à que obtemos no caso da equação do calor. A equação espacial é a mesma, portanto, nos dois casos. A equação para T possui solução,

$$T(t) = a_1 \cos(kvt) + a_2 \,\mathrm{sen}\,(kvt) \,. \tag{6}$$

A constante de separação k está relacionada com as frequências de oscilação. De fato, temos,

$$kv = \omega \,,$$

em que $\omega = 2\pi/T = 2\pi f$ é a frequência angular da onda, f a frequência linear, e T o período (não confundir com a função $T(t)$). Assim,

$$k = \frac{\omega}{v} = \frac{\omega}{\lambda/T} = \frac{\omega T}{\lambda} = \frac{2\pi}{\lambda}\,,$$

em que λ é o comprimento da onda. Vemos que k corresponde ao número de onda. Temos agora que resolver a equação para F.

Considerando $F(x)$

A equação (5) fica,

$$\frac{d^2F}{dx^2} + k^2F = 0\,, \tag{7}$$

com solução,

$$F(x) = a_1 \cos kx + a_2 \operatorname{sen} kx\,. \tag{8}$$

A solução geral é então, usando o princípio de superposição,

$$u(x,t) \;=\; \sum_k [a_1 \cos(kx) + a_2 \operatorname{sen}(kx)][b_1 \cos(kvt) + b_2 \operatorname{sen}(kvt)]\,. \tag{9}$$

Se $k = 0$,

$$F(x) = Ax + B\,. \tag{10}$$

1 Problemas

1. Determine a solução $u(x,t)$ da equação da onda na região aberta (Spiegel [7], probl. 2.32),

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2}\,,$$

$$0 < x < L\,,\quad 0 < t < T\,.$$

As condições inicial e de fronteira são,

$$u(x,0) = \varphi(x)\,,\quad u_t(x,0) = 0\,,$$
$$u(0,t) = u(L,t) = 0\,,$$

que satisfazem as condições de conjunção,

$$\varphi(0) = \varphi(L) = 0\,.$$

A solução é,

$$\begin{aligned} u(x,t) &= Ax + B \\ &+ \sum_j [a_1 \cos(k_j x) + a_2\, sen\,(k_j x)][b_1 \cos(k_j vt) + b_2\, sen\ (k_j vt)]\,. \end{aligned}$$

As condições de contorno e inicial nos dão as equações,

$$\begin{aligned} & u(x,0) = Ax + B \\ & + \sum_j [a_1 \cos(k_j x) + a_2\, sen\,(k_j x)] b_1 = \varphi(x)\,, \\ & u_t(x,0) = \sum_i k_j v [a_1 \cos(k_j x) + a_2\, sen\,(k_j x)] b_2 = 0\,, \\ & u(0,t) = B \\ & + \sum_j a_1 [b_1 \cos(k_j vt) + b_2\, sen\ (k_j vt)] = 0\,, \\ & u(L,t) = AL + B \\ & + \sum_j [a_1 \cos(k_j L) + a_2\, sen\,(k_j L)][b_1 \cos(k_j vt) + b_2\, sen\ (k_j vt)] = 0\,. \end{aligned}$$

Satisfazemos as equações acima escolhendo,

$$\begin{aligned} & A = B = a_1 = b_2 = 0\,, \\ & k_j L = j\pi\,, \quad j = 1, 2, \ldots \end{aligned}$$

A condição para $\varphi(x)$ fica assim,

$$\sum_j b_1 a_2\, sen\,(j\pi x/L) = \varphi(x)\,.$$

Temos uma série de Fourier de senos para $\varphi(x)$, logo,

$$b_1 a_2 = \frac{2}{L} \int_0^L \varphi(x')\, sen\ (j\pi x'/L)\, dx'\,.$$

A solução u é então,

$$u(x,t) = \sum_j sen\,(j\pi x/L)\cos(j\pi vt/L)\frac{2}{L}\int_0^L \varphi(x')\, sen\,(j\pi x'/L)\, dx' \,.$$

Escrevendo a expressão acima em termos de função de Green,

$$u(x,t) = \int_0^L G(x,x',t)\varphi(x')dx' \,,$$

temos,

$$G(x,x',t) = \frac{2}{L}\sum_j \cos(j\pi vt/L)\, sen\,(j\pi x/L)\, sen\,(j\pi x'/L) \,.$$

2. Considere o problema anterior com $u_t(x,0) = \psi(x)$.
A solução é, como antes,

$$\begin{aligned} u(x,t) &= Ax + B \\ &\quad + \sum_j [a_1\cos(k_j x) + a_2\, sen\,(k_j x)][b_1\cos(k_j vt) + b_2\, sen\,(k_j vt)] \,. \end{aligned}$$

As condições de contorno e inicial nos dão as equações,

$$\begin{aligned} & u(x,0) = Ax + B \\ & + \sum_j [a_1\cos(k_j x) + a_2\, sen\,(k_j x)]b_1 = \varphi(x) \,, \\ & u_t(x,0) = \sum_i k_j v[a_1\cos(k_j x) + a_2\, sen\,(k_j x)]b_2 = \psi(x) \,, \\ & u(0,t) = B \\ & + \sum_j a_1[b_1\cos(k_j vt) + b_2\, sen\,(k_j vt)] = 0 \,, \\ & u(L,t) = AL + B \\ & + \sum_j [a_1\cos(k_j L) + a_2\, sen\,(k_j L)][b_1\cos(k_j vt) + b_2\, sen\,(k_j vt)] = 0 \,. \end{aligned}$$

Satisfazemos as equações acima escolhendo,

$$\begin{aligned} & A = B = a_1 = 0 \,, \\ & k_j L = j\pi \,, \quad j = 1, 2, \ldots \end{aligned}$$

As condições para $\varphi(x)$ e $\psi(x)$ ficam assim,

$$\sum_{j=1} a_2 b_1 \, sen\,(j\pi x/L) = \varphi(x)\,,$$
$$\sum_{j=1} k_j v a_2 b_2 \, sen\,(j\pi x/L) = \psi(x)\,.$$

Temos séries de Fourier de senos para $\varphi(x)$ e $\psi(x)$, logo,

$$a_2 b_1 = \frac{2}{L}\int_0^L \varphi(x')\, sen\;(j\pi x'/L)\, dx'$$
$$k_j v a_2 b_2 = \frac{2}{L}\int_0^L \psi(x')\, sen\;(j\pi x'/L)\, dx'\,.$$

A solução u é então,

$$\begin{aligned} u(x,t) \;=\; & \sum_j sen\,(j\pi x/L)\left[\cos(j\pi vt/L)\frac{2}{L}\int_0^L \varphi(x')\, sen\;(j\pi x'/L)\, dx' \right. \\ & \left. + \, sen\;(j\pi vt/L)\frac{2}{j\pi v}\int_0^L \psi(x')\, sen\;(j\pi x'/L)\, dx' \right]\,. \end{aligned}$$

3. Considere o problema 1 com condições de contorno não homogêneas,

$$u(0,t) = u_1\,, \;\; u(L,t) = u_2\,,$$

com u_1 e u_2 constantes.

A solução é,

$$\begin{aligned} u(x,t) \;=\; & Ax + B \\ & + \sum_j [a_1 \cos(k_j x) + a_2\, sen\,(k_j x)][b_1 \cos(k_j vt) + b_2\, sen\;(k_j vt)]\,. \end{aligned}$$

As condições de contorno e inicial nos dão as equações,

$$
\begin{aligned}
&u(x,0) = Ax + B \\
&+ \sum_j [a_1 \cos(k_j x) + a_2\, sen\,(k_j x)] b_1 = \varphi(x)\,, \\
&u_t(x,0) = \sum_i k_j v [a_1 \cos(k_j x) + a_2\, sen\,(k_j x)] b_2 = 0\,, \\
&u(0,t) = B \\
&+ \sum_j a_1 [b_1 \cos(k_j vt) + b_2\, sen\ (k_j vt)] = u_1\,, \\
&u(L,t) = AL + B \\
&+ \sum_j [a_1 \cos(k_j L) + a_2\, sen\,(k_j L)][b_1 \cos(k_j vt) + b_2\, sen\ (k_j vt)] = u_2\,.
\end{aligned}
$$

Satisfazemos as equações acima escolhendo,

$$
\begin{aligned}
&a_1 = b_2 = 0\,, \\
&k_j L = j\pi\,, \quad j = 1, 2, \ldots
\end{aligned}
$$

As condições para $\varphi(x)$, u_1 e u_2 ficam assim,

$$
\begin{aligned}
&u(x,0) = Ax + B + \sum_j a_2 b_1\, sen\,(k_j x) = \varphi(x)\,, \\
&u(0,t) = B = u_1\,, \\
&u(L,t) = AL + B = u_2\,.
\end{aligned}
$$

Para A e B temos,

$$
A = \frac{u_2 - u_1}{L}\,, \quad B = u_1\,.
$$

Para $\varphi(x)$ temos uma série de Fourier de senos, logo,

$$
a_2 b_1 = \frac{2}{L} \int_0^L [\varphi(x') - Ax' - B]\, sen\ (j\pi x'/L)\, dx'\,,
$$

e podemos escrever,

$$
\varphi(x) - Ax - B = \sum_j\ sen\,(j\pi x/L) \frac{2}{L} \int_0^L [\varphi(x') - Ax' - B]\, sen\ (j\pi x'/L)\, dx'\,.
$$

A solução u é então,

$$\begin{aligned} u(x,t) &= \frac{u_2-u_1}{L}x+u_1 \\ &+\sum_j sen\,(k_j x)\cos(k_j vt)\frac{2}{L}\int_0^L [\varphi(x')-Ax'-B]\,sen\,(j\pi x'/L)\,dx'\,. \end{aligned}$$

Calculando as duas últimas integrais (apêndice),

$$\begin{aligned} u(x,t) &= \frac{u_2-u_1}{L}x+u_1 \\ &+\sum_j sen\,(j\pi x/L)\cos(j\pi vt/L)\frac{2}{L}\int_0^L \varphi(x')\,sen\,(j\pi x'/L)\,dx' \\ &-\frac{2(u_2-u_1)}{\pi}\sum_j sen\,(j\pi x/L)\cos(j\pi vt/L)\frac{(-1)^{j+1}}{j} \\ &-\frac{4u_1}{\pi}\sum_j sen\,[(2j-1)\pi x/L]\cos[(2j-1)\pi vt/L]\frac{1}{2j-1}\,. \end{aligned}$$

A solução acima satisfaz as condições em $x=0$ e $x=L$. Em $t=0$,

$$\begin{aligned} u(x,0) &= \frac{u_2-u_1}{L}x+u_1 \\ &+\sum_{j=1} sen\,(j\pi x/L)\frac{2}{L}\int_0^L \varphi(x')\,sen\,(j\pi x'/L)\,dx' \\ &-\frac{2(u_2-u_1)}{\pi}\sum_{j=1} sen\,(j\pi x/L)\frac{(-1)^{j+1}}{j} \\ &-\frac{4u_1}{\pi}\sum_{j=1}\frac{sen\,[(2j-1)\pi x/L]}{2j-1}\,. \end{aligned}$$

Calculando as duas últimas séries (apêndice),

$$\begin{aligned} u(x,0) &= \frac{u_2-u_1}{L}x+u_1 \\ &+\sum_{j=1} sen\,(j\pi x/L)\frac{2}{L}\int_0^L \varphi(x')\,sen\,(j\pi x'/L)\,dx' \\ &-\frac{(u_2-u_1)}{L}x-u_1=\varphi(x)\,, \end{aligned}$$

como esperado. Calculando agora $\partial u/\partial t$,

$$\begin{aligned}\frac{\partial u(x,t)}{\partial t} &= -\sum_j (j\pi v/L)\,sen\,(j\pi x/L)\,sen\,(j\pi vt/L)\frac{2}{L}\int_0^L \varphi(x')\,sen\,(j\pi x'/L)\,dx' \\ &+\frac{2(u_2-u_1)v}{L}\sum_j \,sen\,(j\pi x/L)\,sen\,(j\pi vt/L)(-1)^{j+1} \\ &+\frac{4u_1 v}{L}\sum_j \,sen\,[(2j-1)\pi x/L]\,sen\,[(2j-1)\pi vt/L]\,.\end{aligned}$$

Em $t=0$ temos $\partial u/\partial t = 0$, como esperado.

4. Considere o problema anterior com $u_t(x,0)=\psi(x)$.

A solução é,

$$\begin{aligned}u(x,t) &= Ax+B \\ &+\sum_j [a_1\cos(j\pi x/L)+a_2\,sen\,(k_j x)][b_1\cos(k_j vt)+b_2\,sen\,(k_j vt)]\,.\end{aligned}$$

As condições de contorno e inicial nos dão as equações,

$$\begin{aligned}&u(x,0)=Ax+B \\ &+\sum_j [a_1\cos(k_j x)+a_2\,sen\,(k_j x)]b_1=\varphi(x)\,, \\ &u_t(x,0)=\sum_i k_j v[a_1\cos(k_j x)+a_2\,sen\,(k_j x)]b_2=\psi(x)\,, \\ &u(0,t)=B \\ &+\sum_j a_1[b_1\cos(k_j vt)+b_2\,sen\,(k_j vt)]=u_1\,, \\ &u(L,t)=AL+B \\ &+\sum_j [a_1\cos(k_j L)+a_2\,sen\,(k_j L)][b_1\cos(k_j vt)+b_2\,sen\,(k_j vt)]=u_2\,.\end{aligned}$$

Satisfazemos as equações acima escolhendo,

$$\begin{aligned}&a_1=0\,, \\ &k_j L=j\pi\,,\quad j=1,2,\ldots\end{aligned}$$

As condições para $\varphi(x)$, $\psi(x)$, u_1 e u_2 ficam assim,

$$
\begin{aligned}
&u(x,0) = Ax + B + \sum_j a_2 b_1\, sen\,(k_j x) = \varphi(x)\,,\\
&u_t(x,0) = \sum_i k_j v a_2 b_2\, sen\,(k_j x) = \psi(x)\,,\\
&u(0,t) = B = u_1\,,\\
&u(L,t) = AL + B = u_2\,.
\end{aligned}
$$

Para A e B temos,

$$A = \frac{u_2 - u_1}{L}\,,\ \ B = u_1\,.$$

Para $\varphi(x)$ temos uma série de Fourier de senos, logo,

$$a_2 b_1 = \frac{2}{L}\int_0^L [\varphi(x') - Ax' - B]\, sen\ (j\pi x'/L)\, dx'\,,$$

e podemos escrever,

$$
\begin{aligned}
&\varphi(x) - Ax - B =\\
&= \sum_j\ sen\,(j\pi x/L)\frac{2}{L}\int_0^L [\varphi(x') - Ax' - B]\, sen\ (j\pi x'/L)\, dx'\,.
\end{aligned}
$$

Para $\psi(x)$ também temos uma série de Fourier de senos, assim,

$$k_j v a_2 b_2 = \frac{2}{L}\int_0^L \psi(x')\, sen\ (j\pi x'/L)\, dx'\,,$$

portanto,

$$\psi(x) = \sum_i\ sen\,(k_j x)\frac{2}{L}\int_0^L \psi(x')\, sen\ (j\pi x'/L)\, dx'\,.$$

A solução u é então,

$$
\begin{aligned}
u(x,t) \;\; = \;\; & \frac{u_2 - u_1}{L}x + u_1\\
&+ \sum_j\ sen\,(k_j x) cos(k_j v t)\frac{2}{L}\int_0^L [\varphi(x') - Ax' - B]\, sen\ (j\pi x'/L)\, dx'\\
&+ \sum_j\ sen\,(k_j x)\ sen\ (k_j v t)\frac{2}{j\pi v}\int_0^L \psi(x')\, sen\ (j\pi x'/L)\, dx'
\end{aligned}
$$

Calculando as duas últimas integrais na segunda linha (apêndice),

$$
\begin{aligned}
u(x,t) &= \frac{u_2-u_1}{L}x + u_1 \\
&+ \sum_j sen\,(k_j x) cos(k_j v t) \frac{2}{L}\int_0^L \varphi(x')\, sen\,(j\pi x'/L)\, dx' \\
&- \frac{u_2-u_1}{L}\sum_j sen\,(k_j x) cos(k_j v t)\frac{2L}{j\pi}(-1)^{j+1} \\
&- u_1 \sum_j sen\,[(2j-1)\pi x/L] cos[(2j-1)\pi v t/L]\frac{4}{(2j-1)\pi} \\
&+ \sum_j sen\,(k_j x)\, sen\,(k_j v t)\frac{2}{j\pi v}\int_0^L \psi(x')\, sen\,(j\pi x'/L)\, dx' \,.
\end{aligned}
$$

A solução acima satisfaz as condições em $x = 0$ e $x = L$. Em $t = 0$,

$$
\begin{aligned}
u(x,0) &= \frac{u_2-u_1}{L}x + u_1 \\
&+ \sum_j sen\,(k_j x)\frac{2}{L}\int_0^L \varphi(x')\, sen\,(j\pi x'/L)\, dx' \\
&- \frac{2(u_2-u_1)}{\pi}\sum_j \frac{sen\,(j\pi x/L)}{j}(-1)^{j+1} \\
&- \frac{4u_1}{\pi}\sum_j \frac{sen\,[(2j-1)\pi x/L]}{2j-1} \,.
\end{aligned}
$$

Calculando as duas últimas séries (apêndice),

$$
\begin{aligned}
u(x,0) &= \frac{u_2-u_1}{L}x + u_1 \\
&+ \sum_j sen\,(k_j x)\frac{2}{L}\int_0^L \varphi(x')\, sen\,(j\pi x'/L)\, dx' \\
&- \frac{2(u_2-u_1)}{\pi}\frac{\pi x}{2L} \\
&- \frac{4u_1}{\pi}\frac{\pi}{4} \,, \\
&= \sum_j sen\,(j\pi x/L)\frac{2}{L}\int_0^L \varphi(x')\, sen\,(j\pi x'/L)\, dx' \,, \\
&= \varphi(x) \,,
\end{aligned}
$$

como esperado. Calculando agora $\partial u/\partial t$,

$$\begin{aligned}
\frac{\partial u(x,t)}{\partial t} &= -\sum_j k_j v\, sen\,(k_j x) sen(k_j vt)\frac{2}{L}\int_0^L \varphi(x')\, sen\;(j\pi x'/L)\, dx' \\
&+\frac{u_2-u_1}{L}\sum_j k_j v\, sen\,(k_j x) sen(k_j vt)\frac{2L}{j\pi}(-1)^{j+1} \\
&+u_1\sum_j (4v/L)\, sen\,[(2j-1)\pi x/L] sen[(2j-1)\pi vt/L] \\
&+\sum_j\; sen\,(k_j x) cos\,(k_j vt)\frac{2}{L}\int_0^L \psi(x')\, sen\;(j\pi x'/L)\, dx' \,.
\end{aligned}$$

Em $t=0$,

$$\begin{aligned}
\frac{\partial u(x,0)}{\partial t} &= \sum_j\; sen\,(j\pi x/L)\frac{2}{L}\int_0^L \psi(x')\, sen\;(j\pi x'/L)\, dx' \,, \\
&= \psi(x)\,,
\end{aligned}$$

como esperado.

5. Resolva a equação de onda não homogênea,

$$\frac{\partial^2 u}{\partial t^2} = v^2\frac{\partial^2 u}{\partial x^2} + f(x,t)\,,$$

no intervalo,

$$0 < x < L\,,\;\; t > 0\,,$$

com a condição inicial,

$$u(x,0) = 0\,,\;\; 0 \le x \le L\,,$$

e as condições de contorno homogêneas,

$$u(0,t) = 0\,,\;\; u(L,t) = 0\,,\;\; t \ge 0\,.$$

Expandimos u *e* f *em séries de Fourier de senos,*

$$\begin{aligned}
u(x,t) &= \sum_{i=1} u_i(t)\, sen\;(i\pi x/L)\,, \\
f(x,t) &= \sum_{i=1} f_i(t)\, sen\;(i\pi x/L)\,.
\end{aligned}$$

Temos,

$$u_i(t) = \frac{2}{L}\int_0^L u(x',t)\, sen\ (i\pi x'/L) dx'\,,$$
$$f_i(t) = \frac{2}{L}\int_0^L f(x',t)\, sen\ (i\pi x'/L) dx'\,.$$

Substituindo na equação diferencial temos,

$$\sum_{i=1} \ddot{u}_i(t)\, sen\ (i\pi x/L) =$$
$$= -v^2 \sum_{i=1} (i\pi/L)^2 u_i(t)\, sen\ (i\pi x/L) + \sum_{i=1} f_i(t)\, sen\ (i\pi x/L)\,.$$

Da relação acima obtemos uma equação diferencial linear, de segunda ordem, não homogênea, para $u_i(t)$,

$$\ddot{u}_i(t) + v^2(i\pi/L)^2 u_i(t) = f_i(t)\,.$$

A solução dessa equação é (capítulos. 21 e 22),

$$\begin{aligned}
u_i(t) &= \frac{1}{k_i} sen(k_i t) \int_{t_0}^t cos(k_i t') f_i(t') dt' - \frac{1}{k_i} cos(k_i t) \int_{t_0}^t sen(k_i t') f_i(t') dt'\,,\\
&= \frac{1}{k_i} \int_{t_0}^t \left[sen(k_i t) cos(k_i t') - cos(k_i t) sen(k_i t')\right] f_i(t') dt'\,,\\
&= \frac{1}{k_i} \int_{t_0}^t sen[k(t-t')] f_i(t') dt'\,,
\end{aligned}$$

com $k_i = v(i\pi/L)$ *e* t_0 *qualquer valor em* $t \geq 0$.

A solução u *é então,*

$$\begin{aligned}
u(x,t) &= \sum_{i=1} u_i(t)\, sen\ (i\pi x/L)\,,\\
&= \sum_{i=1} \frac{1}{k_i} \left[sen(k_i t) \int_{t_0}^t cos(k_i t') f_i(t') dt' \right.\\
&\quad \left. - cos(k_i t) \int_{t_0}^t sen(k_i t') f_i(t') dt' \right] sen\ (i\pi x/L)\,,
\end{aligned}$$

Substituindo $f_i(t')$,

$$\begin{aligned} u(x,t) &= \sum_{i=1} \frac{1}{k_i} \left[sen(k_i t) \int_{t_0}^{t} cos(k_i t') f_i(t') dt' \right. \\ & \left. -cos(k_i t) \int_{t_0}^{t} sen(k_i t') f_i(t') dt' \right] sen\,(i\pi x/L)\,, \\ &= \sum_{i=1} \frac{1}{k_i}\, sen\,(i\pi x/L) \times \\ & \times \left[sen(k_i t) \int_{t_0}^{t} cos(k_i t') \frac{2}{L} \int_0^L f(x',t')\, sen\,(i\pi x'/L) dx' dt' \right. \\ & \left. -cos(k_i t) \int_{t_0}^{t} sen(k_i t') \frac{2}{L} \int_0^L f(x',t')\, sen\,(i\pi x'/L) dx' dt' \right]. \end{aligned}$$

Temos $u = 0$ *em* $x = 0$ *e* $x = L$, *como esperado. Em* $t = 0$,

$$\begin{aligned} u(x,0) &= \sum_{i=1} \frac{1}{k_i}\, sen\,(i\pi x/L) \times \\ & \times \left[-\int_{t_0}^{0} sen(k_i t') \frac{2}{L} \int_0^L f(x',t')\, sen\,(i\pi x'/L) dx' dt' \right]. \end{aligned}$$

Temos $u(x,0) = 0$ *escolhendo* $t_0 = 0$, *logo,*

$$\begin{aligned} u(x,t) &= \sum_{i=1} \frac{1}{k_i}\, sen\,(i\pi x/L) \times \\ & \times \left[sen(k_i t) \int_{0}^{t} cos(k_i t') \frac{2}{L} \int_0^L f(x',t')\, sen\,(i\pi x'/L) dx' dt' \right. \\ & \left. -cos(k_i t) \int_{0}^{t} sen(k_i t') \frac{2}{L} \int_0^L f(x',t')\, sen\,(i\pi x'/L) dx' dt' \right]. \end{aligned}$$

Vamos verificar agora que u *satisfaz a equação diferencial. Calculando as derivadas de* u *temos,*

$$
\begin{aligned}
\frac{\partial u(x,t)}{\partial x} &= \sum_{i=1} \frac{(i\pi/L)}{k_i} cos\,(i\pi x/L) \times \\
&\times \left[sen(k_i t) \int_0^t cos(k_i t') \frac{2}{L} \int_0^L f(x',t')\, sen\,(i\pi x'/L) dx' dt' \right. \\
&\left. - cos(k_i t) \int_0^t sen(k_i t') \frac{2}{L} \int_0^L f(x',t')\, sen\,(i\pi x'/L) dx' dt' \right] , \\
\frac{\partial^2 u(x,t)}{\partial x^2} &= -\sum_{i=1} \frac{i\pi}{vL} sen\,(i\pi x/L) \times \\
&\times \left[sen(k_i t) \int_0^t cos(k_i t') \frac{2}{L} \int_0^L f(x',t')\, sen\,(i\pi x'/L) dx' dt' \right. \\
&\left. - cos(k_i t) \int_0^t sen(k_i t') \frac{2}{L} \int_0^L f(x',t')\, sen\,(i\pi x'/L) dx' dt' \right] ,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial u(x,t)}{\partial t} &= \sum_{i=1} sen\,(i\pi x/L) \times \\
&\times \left[cos(k_i t) \int_0^t cos(k_i t') \frac{2}{L} \int_0^L f(x',t')\, sen\,(i\pi x'/L) dx' dt' \right. \\
&\left. + sen(k_i t) \int_0^t sen(k_i t') \frac{2}{L} \int_0^L f(x',t')\, sen\,(i\pi x'/L) dx' dt' \right] , \\
&+ \sum_{i=1} \frac{1}{k_i}\, sen\,(i\pi x/L) \times \\
&\times \left[sen(k_i t) cos(k_i t) \frac{2}{L} \int_0^L f(x',t)\, sen\,(i\pi x'/L) dx' \right. \\
&\left. - cos(k_i t) sen(k_i t) \frac{2}{L} \int_0^L f(x',t)\, sen\,(i\pi x'/L) dx' \right] , \\
&= \sum_{i=1} sen\,(i\pi x/L) \times \\
&\times \left[cos(k_i t) \int_0^t cos(k_i t') \frac{2}{L} \int_0^L f(x',t')\, sen\,(i\pi x'/L) dx' dt' \right. \\
&\left. + sen(k_i t) \int_0^t sen(k_i t') \frac{2}{L} \int_0^L f(x',t')\, sen\,(i\pi x'/L) dx' dt' \right] ,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 u(x,t)}{\partial t^2} &= \sum_{i=1} k_i \, sen\,(i\pi x/L) \times \\
&\times \left[-sen(k_i t) \int_0^t cos(k_i t') \frac{2}{L} \int_0^L f(x',t') \, sen\,(i\pi x'/L) dx' dt' \right. \\
&\left. + cos(k_i t) \int_0^t sen(k_i t') \frac{2}{L} \int_0^L f(x',t') \, sen\,(i\pi x'/L) dx' dt' \right] \\
&+ \sum_{i=1} \, sen\,(i\pi x/L) \times \\
&\times \left[cos(k_i t) cos(k_i t) \frac{2}{L} \int_0^L f(x',t) \, sen\,(i\pi x'/L) dx' \right. \\
&\left. + sen(k_i t) sen(k_i t) \frac{2}{L} \int_0^L f(x',t) \, sen\,(i\pi x'/L) dx' \right] , \\
&= \sum_{i=1} k_i \, sen\,(i\pi x/L) \times \\
&\times \left[-sen(k_i t) \int_0^t cos(k_i t') \frac{2}{L} \int_0^L f(x',t') \, sen\,(i\pi x'/L) dx' dt' \right. \\
&\left. + cos(k_i t) \int_0^t sen(k_i t') \frac{2}{L} \int_0^L f(x',t') \, sen\,(i\pi x'/L) dx' dt' \right] \\
&+ \sum_{i=1} \, sen\,(i\pi x/L) \frac{2}{L} \int_0^L f(x',t) \, sen\,(i\pi x'/L) dx' \, .
\end{aligned}
$$

Portanto,

$$
\begin{aligned}
& \frac{\partial^2 u}{\partial t^2} - v^2 \frac{\partial^2 u}{\partial x^2} = \\
& = \sum_{i=1} k_i \, sen\,(i\pi x/L) \times \\
& \times \left[-sen(k_i t) \int_0^t cos(k_i t') \frac{2}{L} \int_0^L f(x',t') \, sen\,(i\pi x'/L) dx' dt' \right. \\
& \left. + cos(k_i t) \int_0^t sen(k_i t') \frac{2}{L} \int_0^L f(x',t') \, sen\,(i\pi x'/L) dx' dt' \right] \\
& + \sum_{i=1} \, sen\,(i\pi x/L) \frac{2}{L} \int_0^L f(x',t) \, sen\,(i\pi x'/L) dx' \\
& + v^2 \sum_{i=1} \frac{i\pi}{vL} sen\,(i\pi x/L) \times \\
& \times \left[sen(k_i t) \int_0^t cos(k_i t') \frac{2}{L} \int_0^L f(x',t') \, sen\,(i\pi x'/L) dx' dt' \right. \\
& \left. - cos(k_i t) \int_0^t sen(k_i t') \frac{2}{L} \int_0^L f(x',t') \, sen\,(i\pi x'/L) dx' dt' \right] , \\
& = \sum_{i=1} \, sen\,(i\pi x/L) \frac{2}{L} \int_0^L f(x',t) \, sen\,(i\pi x'/L) dx' , \\
= \; & f(x,t) ,
\end{aligned}
$$

como esperado. Notemos que em $t = 0$ *temos* $\partial u/\partial t = 0$.

Podemos escrever u *em termos de uma função de Green. Temos,*

$$
\begin{aligned}
u(x,t) &= \sum_{i=1} \frac{1}{k_i}\, sen\,(i\pi x/L) \times \\
&\times \left[sen(k_i t) \int_0^t cos(k_i t') \frac{2}{L} \int_0^L f(x',t')\, sen\,(i\pi x'/L) dx' dt' \right. \\
&\left. - cos(k_i t) \int_0^t sen(k_i t') \frac{2}{L} \int_0^L f(x',t')\, sen\,(i\pi x'/L) dx' dt' \right] , \\
&= \int_0^t \int_0^L \frac{2}{L} \sum_{i=1} \frac{1}{k_i}\, sen\,(i\pi x/L)\, sen\,(i\pi x'/L) \times \\
&\times \left[sen(k_i t) cos(k_i t') - cos(k_i t) sen(k_i t') \right] f(x',t') dx' dt' , \\
&= \int_0^t \int_0^L \frac{2}{L} \sum_{i=1} \frac{1}{k_i}\, sen\,(i\pi x/L)\, sen\,(i\pi x'/L) \times \\
&\times sen[k_i(t-t')] f(x',t') dx' dt' , \\
&= \int_0^t \int_0^L G(x,x';t,t') f(x',t') dx' dt' ,
\end{aligned}
$$

com

$$
\begin{aligned}
G(x,x';t,t') &= \frac{2}{L} \sum_{i=1} \frac{1}{k_i}\, sen\,(i\pi x/L)\, sen\,(i\pi x'/L) \times \\
&\times sen[k_i(t-t')] , \quad k_i = v i\pi/L \, .
\end{aligned}
$$

Usamos a relação trigonométrica,

$$
sen\,(A \pm B) = \; sen\,A \cos B \pm \cos A\; sen\,B \, ,
$$

6. Considere o problema 5 com condição inicial $u(x,0) = \varphi(x)$ e $u_t(x,0) = 0$.

Temos agora, considerando também a solução da equação homogênea para u_i*,*

$$
\begin{aligned}
u_i(t) &= c_1 sen(k_i t) + c_2 cos(k_i t) \\
&+ \frac{1}{k_i} sen(k_i t) \int_0^t cos(k_i t') f_i(t') dt' \\
&- \frac{1}{k_i} cos(k_i t) \int_0^t sen(k_i t') f_i(t') dt' ,
\end{aligned}
$$

com $k_i = v(i\pi/L)$ *e,*

$$f_i(t) = \frac{2}{L}\int_0^L f(x',t)\,sen\,(i\pi x'/L)dx'\,.$$

A solução u é então,

$$\begin{aligned} u(x,t) &= \sum_{i=1} u_i(t)\,sen\,(i\pi x/L)\,, \\ &= \sum_{i=1} sen\,(i\pi x/L)\times \\ &\quad \times\Big[c_1 sen(k_i t) + c_2 cos(k_i t) \\ &\quad +\frac{1}{k_i} sen(k_i t)\int_0^t cos(k_i t')f_i(t')dt' \\ &\quad -\frac{1}{k_i} cos(k_i t)\int_0^t sen(k_i t')f_i(t')dt'\Big]\,. \end{aligned}$$

Em $x = 0$ *e* $x = L$ *temos* $u = 0$ *como esperado. Calculando* $\partial u/\partial t$*,*

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} &= \sum_{i=1} u_i(t)\,sen\,(i\pi x/L)\,, \\ &= \sum_{i=1} sen\,(i\pi x/L)\times \\ &\quad \times\Big[c_1 k_i cos(k_i t) - c_2 k_i sen(k_i t) \\ &\quad +cos(k_i t)\int_0^t cos(k_i t')f_i(t')dt' \\ &\quad +sen(k_i t)\int_0^t sen(k_i t')f_i(t')dt' \\ &\quad +\frac{1}{k_i} sen(k_i t)cos(k_i t)f_i(t) \\ &\quad -\frac{1}{k_i} cos(k_i t)sen(k_i t)f_i(t)\Big]\,, \\ &= \sum_{i=1} sen\,(i\pi x/L)\times \\ &\quad \times\Big[c_1 k_i cos(k_i t) - c_2 k_i sen(k_i t) \\ &\quad +cos(k_i t)\int_0^t cos(k_i t')f_i(t')dt' \\ &\quad +sen(k_i t)\int_0^t sen(k_i t')f_i(t')dt'\Big]\,. \end{aligned}$$

Em $t = 0$,

$$\frac{\partial u(x,0)}{\partial t} = \sum_{i=1} c_1 k_i \, sen\,(i\pi x/L)\,.$$

Portanto $u_t(x,0) = 0$ *se* $c_1 = 0$, *logo,*

$$\begin{aligned} u(x,t) &= \sum_{i=1} sen\,(i\pi x/L)\,[c_2 cos(k_i t) \\ &\quad + \frac{1}{k_i} sen(k_i t) \int_0^t cos(k_i t') f_i(t') dt' \\ &\quad - \frac{1}{k_i} cos(k_i t) \int_0^t sen(k_i t') f_i(t') dt' \Big]\,. \end{aligned}$$

Em $t = 0$,

$$u(x,0) = \sum_{i=1} c_2 \, sen\,(i\pi x/L) = \varphi(x)\,.$$

Temos uma série de Fourier de senos para $\varphi(x)$, *logo,*

$$c_2 = \frac{2}{L}\int_0^L \varphi(x)\, sen\,(i\pi x/L)\, dx\,.$$

A expansão de $\varphi(x)$ *é então,*

$$\varphi(x) = \sum_{i=1} sen\,(i\pi x/L) \frac{2}{L}\int_0^L \varphi(x')\, sen\,(i\pi x'/L)\, dx'\,.$$

A solução u *é assim,*

$$\begin{aligned} u(x,t) &= \sum_{i=1} sen\,(i\pi x/L) \times \\ &\quad \times \Big[cos(k_i t) \frac{2}{L}\int_0^L \varphi(x')\, sen\,(i\pi x'/L)\, dx' \\ &\quad + \frac{1}{k_i} sen(k_i t) \int_0^t cos(k_i t') f_i(t') dt' \\ &\quad - \frac{1}{k_i} cos(k_i t) \int_0^t sen(k_i t') f_i(t') dt' \Big]\,, \quad k_i = v(i\pi/L)\,. \end{aligned}$$

A solução acima satisfaz as condições em $x = 0$, $x = L$ e $t = 0$. Calculando as derivadas de u,

$$
\begin{aligned}
\frac{\partial u(x,t)}{\partial t} &= \sum_{i=1} sen\ (i\pi x/L) \times \\
&\times \left[-k_i sen(k_i t) \frac{2}{L} \int_0^L \varphi(x')\, sen\ (i\pi x'/L)\, dx' \right. \\
&+ cos(k_i t) \int_0^t cos(k_i t') f_i(t') dt' \\
&+ sen(k_i t) \int_0^t sen(k_i t') f_i(t') dt' \\
&+ \frac{1}{k_i} sen(k_i t) cos(k_i t) f_i(t) \\
&\left. - \frac{1}{k_i} cos(k_i t) sen(k_i t) f_i(t) \right], \\
&= \sum_{i=1} sen\ (i\pi x/L) \times \\
&\times \left[-k_i sen(k_i t) \frac{2}{L} \int_0^L \varphi(x')\, sen\ (i\pi x'/L)\, dx' \right. \\
&+ cos(k_i t) \int_0^t cos(k_i t') f_i(t') dt' \\
&\left. + sen(k_i t) \int_0^t sen(k_i t') f_i(t') dt' \right].
\end{aligned}
$$

Em $t = 0$ temos $u_t = 0$, como esperado. Prosseguindo com o cálculo das derivadas,

$$
\begin{aligned}
\frac{\partial^2 u(x,t)}{\partial t^2} &= \sum_{i=1} sen\ (i\pi x/L) \times \\
&\times \left[-k_i^2 cos(k_i t) \frac{2}{L} \int_0^L \varphi(x')\, sen\ (i\pi x'/L)\, dx' \right. \\
&-k_i sen(k_i t) \int_0^t cos(k_i t') f_i(t') dt' \\
&+k_i cos(k_i t) \int_0^t sen(k_i t') f_i(t') dt' \\
&+cos(k_i t) cos(k_i t) f_i(t) \\
&+sen(k_i t) sen(k_i t) f_i(t)] \ , \\
&= \sum_{i=1} sen\ (i\pi x/L) \times \\
&\times \left[-k_i^2 cos(k_i t) \frac{2}{L} \int_0^L \varphi(x')\, sen\ (i\pi x'/L)\, dx' \right. \\
&-k_i sen(k_i t) \int_0^t cos(k_i t') f_i(t') dt' \\
&+k_i cos(k_i t) \int_0^t sen(k_i t') f_i(t') dt' \\
&+f_i(t)] \ .
\end{aligned}
$$

Derivando agora em relação a x,

$$
\begin{aligned}
\frac{\partial u(x,t)}{\partial x} &= \sum_{i=1} (i\pi/L) cos\,(i\pi x/L) \times \\
&\times \left[cos(k_i t) \frac{2}{L} \int_0^L \varphi(x')\; sen\; (i\pi x'/L)\, dx' \right. \\
&+ \frac{1}{k_i} sen(k_i t) \int_0^t cos(k_i t') f_i(t') dt' \\
&\left. - \frac{1}{k_i} cos(k_i t) \int_0^t sen(k_i t') f_i(t') dt' \right] , \\
\frac{\partial^2 u(x,t)}{\partial x^2} &= - \sum_{i=1} (i\pi/L)^2 sen\,(i\pi x/L) \times \\
&\times \left[cos(k_i t) \frac{2}{L} \int_0^L \varphi(x')\; sen\; (i\pi x'/L)\, dx' \right. \\
&+ \frac{1}{k_i} sen(k_i t) \int_0^t cos(k_i t') f_i(t') dt' \\
&\left. - \frac{1}{k_i} cos(k_i t) \int_0^t sen(k_i t') f_i(t') dt' \right] .
\end{aligned}
$$

Portanto,

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} - v^2 \frac{\partial^2 u}{\partial x^2} = \\
&= \sum_{i=1} sen\,(i\pi x/L) \times \\
&\times \left[-k_i^2 cos(k_i t) \frac{2}{L} \int_0^L \varphi(x')\, sen\,(i\pi x'/L)\, dx' \right. \\
&-k_i sen(k_i t) \int_0^t cos(k_i t') f_i(t') dt' \\
&+k_i cos(k_i t) \int_0^t sen(k_i t') f_i(t') dt' \\
&+f_i(t)] \\
&+v^2 \sum_{i=1} (i\pi/L)^2 sen\,(i\pi x/L) \times \\
&\times \left[cos(k_i t) \frac{2}{L} \int_0^L \varphi(x')\, sen\,(i\pi x'/L)\, dx' \right. \\
&+\frac{1}{k_i} sen(k_i t) \int_0^t cos(k_i t') f_i(t') dt' \\
&\left. -\frac{1}{k_i} cos(k_i t) \int_0^t sen(k_i t') f_i(t') dt' \right], \\
&= \sum_{i=1} sen\,(i\pi x/L) f_i(t) = f(x,t)\,,
\end{aligned}
$$

como esperado. Lembrando que a expansão de f é,

$$
\begin{aligned}
f(x,t) &= \sum_{i=1} f_i(t)\, sen\,(i\pi x/L)\,, \\
&= \sum_{i=1} sen\,(i\pi x/L) \frac{2}{L} \int_0^L f(x',t)\, sen\,(i\pi x'/L) dx'\,,
\end{aligned}
$$

pois,

$$
f_i(t) = \frac{2}{L} \int_0^L f(x',t)\, sen\,(i\pi x'/L) dx'\,.
$$

7. Considere o problema 5 com condições iniciais $u(x,0) = \varphi(x)$ e $u_t(x,0) = \psi(x)$.

Temos, como no problema 6,

$$
\begin{aligned}
u_i(t) &= c_1 sen(k_i t) + c_2 cos(k_i t) \\
&+\frac{1}{k_i} sen(k_i t) \int_0^t cos(k_i t') f_i(t') dt' \\
&-\frac{1}{k_i} cos(k_i t) \int_0^t sen(k_i t') f_i(t') dt' \,,
\end{aligned}
$$

com $k_i = v(i\pi/L)$ *e,*

$$
f_i(t) = \frac{2}{L} \int_0^L f(x', t)\, sen\ (i\pi x'/L) dx' \,.
$$

A solução u é então,

$$
\begin{aligned}
u(x,t) &= \sum_{i=1} u_i(t)\, sen\ (i\pi x/L)\,, \\
&= \sum_{i=1} sen\ (i\pi x/L) \times \\
&\times \left[c_1 sen(k_i t) + c_2 cos(k_i t) \right. \\
&+\frac{1}{k_i} sen(k_i t) \int_0^t cos(k_i t') f_i(t') dt' \\
&\left. -\frac{1}{k_i} cos(k_i t) \int_0^t sen(k_i t') f_i(t') dt' \right] \,.
\end{aligned}
$$

Em $x = 0$ *e* $x = L$ *temos* $u = 0$ *como esperado. Calculando* $\partial u/\partial t$*,*

$$
\begin{aligned}
\frac{\partial u(x,t)}{\partial t} &= \sum_{i=1} u_i(t)\, sen\, (i\pi x/L)\,, \\
&= \sum_{i=1} sen\, (i\pi x/L) \times \\
&\quad \times [c_1 k_i cos(k_i t) - c_2 k_i sen(k_i t) \\
&\quad + cos(k_i t) \int_0^t cos(k_i t') f_i(t') dt' \\
&\quad + sen(k_i t) \int_0^t sen(k_i t') f_i(t') dt' \\
&\quad + \frac{1}{k_i} sen(k_i t) cos(k_i t) f_i(t) \\
&\quad \left. - \frac{1}{k_i} cos(k_i t) sen(k_i t) f_i(t) \right]\,, \\
&= \sum_{i=1} sen\, (i\pi x/L) \times \\
&\quad \times [c_1 k_i cos(k_i t) - c_2 k_i sen(k_i t) \\
&\quad + cos(k_i t) \int_0^t cos(k_i t') f_i(t') dt' \\
&\quad \left. + sen(k_i t) \int_0^t sen(k_i t') f_i(t') dt' \right]\,.
\end{aligned}
$$

Em $t = 0$*,*

$$
\begin{aligned}
u(x,0) &= \sum_{i=1} c_2\, sen\, (i\pi x/L) = \varphi(x)\,, \\
\frac{\partial u(x,0)}{\partial t} &= \sum_{i=1} c_1 k_i\, sen\, (i\pi x/L) = \psi(x)\,.
\end{aligned}
$$

Temos então séries de Fourier de senos para $\varphi(x)$ *e* $\psi(x)$*, logo,*

$$
\begin{aligned}
c_2 &= \frac{2}{L} \int_0^L \varphi(x)\, sen\, (i\pi x/L)\, dx\,, \\
c_1 k_i &= \frac{2}{L} \int_0^L \psi(x)\, sen\, (i\pi x/L)\, dx\,.
\end{aligned}
$$

As expansões de $\varphi(x)$ *e* $\psi(x)$ *são então,*

$$
\varphi(x) = \sum_{i=1} sen\ (i\pi x/L) \frac{2}{L} \int_0^L \varphi(x')\, sen\ (i\pi x'/L)\, dx' \,,
$$
$$
\psi(x) = \sum_{i=1} sen\ (i\pi x/L) \frac{2}{L} \int_0^L \psi(x')\, sen\ (i\pi x'/L)\, dx' \,.
$$

A solução u é assim,

$$
\begin{aligned}
u(x,t) \;\; = \;\; & \sum_{i=1} sen\ (i\pi x/L) \times \\
& \times \Bigg[sen(k_i t) \frac{1}{k_i} \frac{2}{L} \int_0^L \psi(x')\, sen\ (i\pi' x/L)\, dx' \\
& + cos(k_i t) \frac{2}{L} \int_0^L \varphi(x')\, sen\ (i\pi x'/L)\, dx' \\
& + \frac{1}{k_i} sen(k_i t) \int_0^t cos(k_i t') f_i(t') dt' \\
& - \frac{1}{k_i} cos(k_i t) \int_0^t sen(k_i t') f_i(t') dt' \Bigg] \,, \;\; k_i = v(i\pi/L) \,.
\end{aligned}
$$

A solução acima satisfaz as condições em $x = 0$, $x = L$ e $t = 0$. Calculando as derivadas de u,

$$
\begin{aligned}
\frac{\partial u(x,t)}{\partial t} &= \sum_{i=1} sen\ (i\pi x/L) \times \\
& \times \left[k_i cos(k_i t) \frac{1}{k_i} \frac{2}{L} \int_0^L \psi(x')\, sen\ (i\pi' x/L)\, dx' \right. \\
& -k_i sen(k_i t) \frac{2}{L} \int_0^L \varphi(x')\, sen\ (i\pi x'/L)\, dx' \\
& +cos(k_i t) \int_0^t cos(k_i t') f_i(t') dt' \\
& +sen(k_i t) \int_0^t sen(k_i t') f_i(t') dt' \\
& +\frac{1}{k_i} sen(k_i t) cos(k_i t) f_i(t) \\
& \left. -\frac{1}{k_i} cos(k_i t) sen(k_i t) f_i(t) \right] , \\
&= \sum_{i=1} sen\ (i\pi x/L) \times \\
& \times \left[k_i cos(k_i t) \frac{1}{k_i} \frac{2}{L} \int_0^L \psi(x')\, sen\ (i\pi' x/L)\, dx' \right. \\
& -k_i sen(k_i t) \frac{2}{L} \int_0^L \varphi(x')\, sen\ (i\pi x'/L)\, dx' \\
& +cos(k_i t) \int_0^t cos(k_i t') f_i(t') dt' \\
& \left. +sen(k_i t) \int_0^t sen(k_i t') f_i(t') dt' \right] .
\end{aligned}
$$

Em $t = 0$,

$$
\frac{\partial u(x,0)}{\partial t} = \sum_{i=1} sen\ (i\pi x/L) \frac{2}{L} \int_0^L \psi(x')\, sen\ (i\pi' x/L)\, dx' = \psi(x) \, ,
$$

como esperado. Prosseguindo com o cálculo das derivadas,

$$
\begin{aligned}
\frac{\partial^2 u(x,t)}{\partial t^2} &= \sum_{i=1} sen\ (i\pi x/L) \times \\
&\times \left[-k_i sen(k_i t)\frac{2}{L}\int_0^L \psi(x')\, sen\ (i\pi' x/L)\, dx' \right. \\
&-k_i^2 cos(k_i t)\frac{2}{L}\int_0^L \varphi(x')\, sen\ (i\pi x'/L)\, dx' \\
&-k_i sen(k_i t)\int_0^t cos(k_i t') f_i(t') dt' \\
&+k_i cos(k_i t)\int_0^t sen(k_i t') f_i(t') dt' \\
&+cos(k_i t) cos(k_i t) f_i(t) \\
&+sen(k_i t) sen(k_i t) f_i(t)] \ , \\
&= \sum_{i=1} sen\ (i\pi x/L) \times \\
&\times \left[-k_i sen(k_i t)\frac{2}{L}\int_0^L \psi(x')\, sen\ (i\pi' x/L)\, dx' \right. \\
&-k_i^2 cos(k_i t)\frac{2}{L}\int_0^L \varphi(x')\, sen\ (i\pi x'/L)\, dx' \\
&-k_i sen(k_i t)\int_0^t cos(k_i t') f_i(t') dt' \\
&+k_i cos(k_i t)\int_0^t sen(k_i t') f_i(t') dt' \\
&+f_i(t)] \ .
\end{aligned}
$$

Derivando agora em relação a x,

$$
\begin{aligned}
\frac{\partial u(x,t)}{\partial x} &= \sum_{i=1}(i\pi/L)cos\,(i\pi x/L)\times \\
&\times\left[sen(k_it)\frac{1}{k_i}\frac{2}{L}\int_0^L \psi(x')\,sen\;(i\pi' x/L)\,dx'\right. \\
&+cos(k_it)\frac{2}{L}\int_0^L \varphi(x')\,sen\;(i\pi x'/L)\,dx' \\
&+\frac{1}{k_i}sen(k_it)\int_0^t cos(k_it')f_i(t')dt' \\
&\left.-\frac{1}{k_i}cos(k_it)\int_0^t sen(k_it')f_i(t')dt'\right]\,, \\
\frac{\partial^2 u(x,t)}{\partial x^2} &= -\sum_{i=1}(i\pi/L)^2 sen\,(i\pi x/L)\times \\
&\times\left[sen(k_it)\frac{1}{k_i}\frac{2}{L}\int_0^L \psi(x')\,sen\;(i\pi' x/L)\,dx'\right. \\
&+cos(k_it)\frac{2}{L}\int_0^L \varphi(x')\,sen\;(i\pi x'/L)\,dx' \\
&+\frac{1}{k_i}sen(k_it)\int_0^t cos(k_it')f_i(t')dt' \\
&\left.-\frac{1}{k_i}cos(k_it)\int_0^t sen(k_it')f_i(t')dt'\right]\,.
\end{aligned}
$$

Portanto,

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} - v^2 \frac{\partial^2 u}{\partial x^2} = \\
&= \sum_{i=1} sen\,(i\pi x/L) \times \\
&\times \left[-k_i sen(k_i t) \frac{2}{L} \int_0^L \psi(x')\, sen\,(i\pi' x/L)\, dx' \right. \\
&-k_i^2 cos(k_i t) \frac{2}{L} \int_0^L \varphi(x')\, sen\,(i\pi x'/L)\, dx' \\
&-k_i sen(k_i t) \int_0^t cos(k_i t') f_i(t') dt' \\
&+k_i cos(k_i t) \int_0^t sen(k_i t') f_i(t') dt' \\
&+f_i(t)] \\
&+v^2 \sum_{i=1} (i\pi/L)^2 sen\,(i\pi x/L) \times \\
&\times \left[sen(k_i t) \frac{1}{k_i} \frac{2}{L} \int_0^L \psi(x')\, sen\,(i\pi' x/L)\, dx' \right. \\
&+cos(k_i t) \frac{2}{L} \int_0^L \varphi(x')\, sen\,(i\pi x'/L)\, dx' \\
&+\frac{1}{k_i} sen(k_i t) \int_0^t cos(k_i t') f_i(t') dt' \\
&\left. -\frac{1}{k_i} cos(k_i t) \int_0^t sen(k_i t') f_i(t') dt' \right], \\
&= \sum_{i=1} sen\,(i\pi x/L) f_i(t) = f(x,t)\,,
\end{aligned}
$$

como esperado.

8. Considere o problema 5 com condições de contorno constantes $u(0,t) = u_1$, $u(L,t) = u_2$.

Temos agora,

$$\begin{aligned}
u(x,t) &= \frac{u_2-u_1}{L}x+u_1 \\
&+\sum_{i=1} sen\;(i\pi x/L)\times \\
&\times\left[c_1 sen(k_i t)+c_2 cos(k_i t)\right. \\
&+\frac{1}{k_i}sen(k_i t)\int_0^t cos(k_i t')f_i(t')dt' \\
&\left.-\frac{1}{k_i}cos(k_i t)\int_0^t sen(k_i t')f_i(t')dt'\right]\;,\quad k_i=vi\pi/L\,.
\end{aligned}$$

Calculando $\partial u/\partial t$,

$$\begin{aligned}
\frac{\partial u(x,t)}{\partial t} &= \sum_{i=1} sen\;(i\pi x/L)\times \\
&\times\left[c_1 k_i cos(k_i t)-c_2 k_i sen(k_i t)\right. \\
&+cos(k_i t)\int_0^t cos(k_i t')f_i(t')dt' \\
&+sen(k_i t)\int_0^t sen(k_i t')f_i(t')dt' \\
&+\frac{1}{k_i}sen(k_i t)cos(k_i t)f_i(t) \\
&\left.-\frac{1}{k_i}cos(k_i t)sen(k_i t)f_i(t)\right]\;, \\
&= \sum_{i=1} sen\;(i\pi x/L)\times \\
&\times\left[c_1 k_i cos(k_i t)-c_2 k_i sen(k_i t)\right. \\
&+cos(k_i t)\int_0^t cos(k_i t')f_i(t')dt' \\
&\left.+sen(k_i t)\int_0^t sen(k_i t')f_i(t')dt'\right]\;.
\end{aligned}$$

As condições de contorno e iniciais nos dão as equações,

$$
\begin{aligned}
u(0,t) &= u_1\,,\\
u(L,t) &= u_2\,,\\
u(x,0) &= \frac{u_2-u_1}{L}x+u_1+\sum_{i=1} c_2\, sen\,(i\pi x/L)=0\,,\\
\frac{\partial u(x,0)}{\partial t} &= \sum_{i=1} c_1 k_i\, sen\,(i\pi x/L)=0\,.
\end{aligned}
$$

Portanto $c_1=0$ *e,*

$$
-\frac{u_2-u_1}{L}x-u_1=\sum_{i=1} c_2\, sen\,(i\pi x/L)\,.
$$

Temos uma série de Fourier de senos, logo,

$$
c_2=\frac{2}{L}\int_0^L\left(-\frac{u_2-u_1}{L}x-u_1\right)\, sen\,(i\pi x/L)\,dx\,,
$$

portanto,

$$
\begin{aligned}
&-\frac{u_2-u_1}{L}x-u_1=\\
&=\sum_{i=1} sen\,(i\pi x/L)\frac{2}{L}\int_0^L\left(-\frac{u_2-u_1}{L}x'-u_1\right)\, sen\,(i\pi x'/L)\,dx'\,.
\end{aligned}
$$

A solução u é então,

$$
\begin{aligned}
u(x,t) &= \frac{u_2-u_1}{L}x+u_1\\
&+\sum_{i=1} sen\,(i\pi x/L)\times\\
&\times\left[cos(k_i t)\frac{2}{L}\int_0^L\left(-\frac{u_2-u_1}{L}x'-u_1\right)\, sen\,(i\pi x'/L)\,dx'\right.\\
&+\frac{1}{k_i}sen(k_i t)\int_0^t cos(k_i t')f_i(t')dt'\\
&\left.-\frac{1}{k_i}cos(k_i t)\int_0^t sen(k_i t')f_i(t')dt'\right]\,,\quad k_i=vi\pi/L\,.
\end{aligned}
$$

Calculando $\partial u/\partial t$*,*

$$
\begin{aligned}
\frac{\partial u(x,t)}{\partial t} &= \sum_{i=1} sen\ (i\pi x/L) \times \\
&\times \left[-k_i sen(k_i t) \frac{2}{L} \int_0^L \left(-\frac{u_2-u_1}{L} x' - u_1 \right) sen\ (i\pi x'/L)\, dx' \right. \\
&+ cos(k_i t) \int_0^t cos(k_i t') f_i(t') dt' \\
&+ sen(k_i t) \int_0^t sen(k_i t') f_i(t') dt' \\
&+ \frac{1}{k_i} sen(k_i t) cos(k_i t) f_i(t) \\
&\left. - \frac{1}{k_i} cos(k_i t) sen(k_i t) f_i(t) \right] , \\
&= \sum_{i=1} sen\ (i\pi x/L) \times \\
&\times \left[-k_i sen(k_i t) \frac{2}{L} \int_0^L \left(-\frac{u_2-u_1}{L} x' - u_1 \right) sen\ (i\pi x'/L)\, dx' \right. \\
&+ cos(k_i t) \int_0^t cos(k_i t') f_i(t') dt' \\
&\left. + sen(k_i t) \int_0^t sen(k_i t') f_i(t') dt' \right] .
\end{aligned}
$$

Temos $u(0,t) = u_1$ *e* $u(L,t) = u_2$. *Em* $t = 0$,

$$
\begin{aligned}
u(x,0) &= \frac{u_2-u_1}{L} x + u_1 \\
&+ \sum_{i=1} sen\ (i\pi x/L) \times \\
&\times \left[\frac{2}{L} \int_0^L \left(-\frac{u_2-u_1}{L} x' - u_1 \right) sen\ (i\pi x'/L)\, dx' \right] = 0 , \\
\frac{\partial u(x,0)}{\partial t} &= 0 ,
\end{aligned}
$$

como esperado. Vamos verificar agora que u *satisfaz a equação diferencial. Calculando as derivadas de* u,

$$
\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= \sum_{i=1} sen\ (i\pi x/L) \times \\
&\times \left[-k_i^2 cos(k_i t) \frac{2}{L} \int_0^L \left(-\frac{u_2 - u_1}{L} x' - u_1 \right) sen\ (i\pi x'/L)\, dx' \right. \\
&-k_i sen(k_i t) \int_0^t cos(k_i t') f_i(t') dt' \\
&+k_i cos(k_i t) \int_0^t sen(k_i t') f_i(t') dt' \\
&+cos(k_i t) cos(k_i t) f_i(t) \\
&+sen(k_i t) sen(k_i t) f_i(t)] \ , \\
&= \sum_{i=1} sen\ (i\pi x/L) \times \\
&\times \left[-k_i^2 cos(k_i t) \frac{2}{L} \int_0^L \left(-\frac{u_2 - u_1}{L} x' - u_1 \right) sen\ (i\pi x'/L)\, dx' \right. \\
&-k_i sen(k_i t) \int_0^t cos(k_i t') f_i(t') dt' \\
&+k_i cos(k_i t) \int_0^t sen(k_i t') f_i(t') dt' \\
&+f_i(t)] \ ,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{u_2 - u_1}{L} \\
&+ \sum_{i=1} (i\pi/L) cos\,(i\pi x/L) \times \\
&\times \left[cos(k_i t) \frac{2}{L} \int_0^L \left(-\frac{u_2 - u_1}{L} x' - u_1 \right) sen\ (i\pi x'/L)\, dx' \right. \\
&+ \frac{1}{k_i} sen(k_i t) \int_0^t cos(k_i t') f_i(t') dt' \\
&\left. - \frac{1}{k_i} cos(k_i t) \int_0^t sen(k_i t') f_i(t') dt' \right] , \\
\frac{\partial^2 u}{\partial x^2} &= -\sum_{i=1} (i\pi/L)^2 sen\,(i\pi x/L) \times \\
&\times \left[cos(k_i t) \frac{2}{L} \int_0^L \left(-\frac{u_2 - u_1}{L} x' - u_1 \right) sen\ (i\pi x'/L)\, dx' \right. \\
&+ \frac{1}{k_i} sen(k_i t) \int_0^t cos(k_i t') f_i(t') dt' \\
&\left. - \frac{1}{k_i} cos(k_i t) \int_0^t sen(k_i t') f_i(t') dt' \right] .
\end{aligned}
$$

Lembremos que $k_i = vi\pi/L$. *Portanto,*

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} - v^2 \frac{\partial^2 u}{\partial x^2} = \\
&= \sum_{i=1} sen\,(i\pi x/L) \times \\
&\times \left[-k_i^2 cos(k_i t) \frac{2}{L} \int_0^L \left(-\frac{u_2 - u_1}{L} x' - u_1 \right) sen\,(i\pi x'/L)\, dx' \right. \\
&-k_i sen(k_i t) \int_0^t cos(k_i t') f_i(t') dt' \\
&+k_i cos(k_i t) \int_0^t sen(k_i t') f_i(t') dt' \\
&+f_i(t)] \\
&+v^2 \sum_{i=1} (i\pi/L)^2 sen\,(i\pi x/L) \times \\
&\times \left[cos(k_i t) \frac{2}{L} \int_0^L \left(-\frac{u_2 - u_1}{L} x' - u_1 \right) sen\,(i\pi x'/L)\, dx' \right. \\
&+\frac{1}{k_i} sen(k_i t) \int_0^t cos(k_i t') f_i(t') dt' \\
&\left. -\frac{1}{k_i} cos(k_i t) \int_0^t sen(k_i t') f_i(t') dt' \right], \\
&= \sum_{i=1} sen\,(i\pi x/L) f_i(t) =
\end{aligned}
$$

Lembrando que a expansão de f é,

$$
\begin{aligned}
f(x,t) &= \sum_{i=1} f_i(t)\, sen\,(i\pi x/L)\,, \\
&= \sum_{i=1} sen\,(i\pi x/L) \frac{2}{L} \int_0^L f(x',t)\, sen\,(i\pi x'/L) dx'\,,
\end{aligned}
$$

pois,

$$
f_i(t) = \frac{2}{L} \int_0^L f(x',t)\, sen\,(i\pi x'/L) dx'\,.
$$

9. Considere o problema 5 com condições de contorno constantes $u(0,t) = u_1$, $u(L,t) = u_2$, e condições iniciais $u(x,0) = \varphi(x)$, $u_t(x,0) = \psi(x)$.

Escrevemos a solução como,

$$
\begin{aligned}
u(x,t) &= \frac{u_2 - u_1}{L} x + u_1 \\
&+ \sum_{i=1} sen\ (i\pi x/L) \times \\
&\times \left[c_1 sen(k_i t) + c_2 cos(k_i t) \right. \\
&+ \frac{1}{k_i} sen(k_i t) \int_0^t cos(k_i t') f_i(t') dt' \\
&\left. - \frac{1}{k_i} cos(k_i t) \int_0^t sen(k_i t') f_i(t') dt' \right], \quad k_i = v i\pi/L \, .
\end{aligned}
$$

Calculando $\partial u/\partial t$,

$$
\begin{aligned}
\frac{\partial u(x,t)}{\partial t} &= \sum_{i=1} sen\ (i\pi x/L) \times \\
&\times \left[c_1 k_i cos(k_i t) - c_2 k_i sen(k_i t) \right. \\
&+ cos(k_i t) \int_0^t cos(k_i t') f_i(t') dt' \\
&+ sen(k_i t) \int_0^t sen(k_i t') f_i(t') dt' \\
&+ \frac{1}{k_i} sen(k_i t) cos(k_i t) f_i(t) \\
&\left. - \frac{1}{k_i} cos(k_i t) sen(k_i t) f_i(t) \right], \\
&= \sum_{i=1} sen\ (i\pi x/L) \times \\
&\times \left[c_1 k_i cos(k_i t) - c_2 k_i sen(k_i t) \right. \\
&+ cos(k_i t) \int_0^t cos(k_i t') f_i(t') dt' \\
&\left. + sen(k_i t) \int_0^t sen(k_i t') f_i(t') dt' \right] .
\end{aligned}
$$

As condições de contorno e iniciais nos dão as equações,

$$
\begin{aligned}
u(0,t) &= u_1\,, \\
u(L,t) &= u_2\,, \\
u(x,0) &= \frac{u_2-u_1}{L}x+u_1 \\
&\quad +\sum_{i=1} c_2 sen\,(i\pi x/L) = \varphi(x)\,, \\
\frac{\partial u(x,0)}{\partial t} &= \sum_{i=1} c_1 k_i\, sen\,(i\pi x/L) = \psi(x)\,.
\end{aligned}
$$

As duas últimas expressões são séries de Fourier de senos,

$$
\begin{aligned}
&\varphi(x)-\frac{u_2-u_1}{L}x-u_1 = \sum_{i=1} c_2 sen\,(i\pi x/L)\,, \\
&\psi(x) = \sum_{i=1} c_1 k_i\, sen\,(i\pi x/L)\,,
\end{aligned}
$$

logo,

$$
\begin{aligned}
&c_2 = \frac{2}{L}\int_0^L \left(\varphi(x)-\frac{u_2-u_1}{L}x-u_1\right)\, sen\,(i\pi x/L)\,dx\,, \\
&c_1 k_i = \frac{2}{L}\int_0^L \psi(x)\, sen\,(i\pi x/L)\,dx\,.
\end{aligned}
$$

Temos assim as séries,

$$
\begin{aligned}
&\varphi(x)-\frac{u_2-u_1}{L}x-u_1 = \\
&=\sum_{i=1} sen\,(i\pi x/L)\frac{2}{L}\int_0^L \left(\varphi(x')-\frac{u_2-u_1}{L}x'-u_1\right)\, sen\,(i\pi x'/L)\,dx'\,, \\
&\psi(x) = \sum_{i=1}\, sen\,(i\pi x/L)\frac{2}{L}\int_0^L \psi(x')\, sen\,(i\pi x'/L)\,dx'\,.
\end{aligned}
$$

A solução fica portanto,

$$
\begin{aligned}
u(x,t) &= \frac{u_2-u_1}{L}x+u_1 \\
&+\sum_{i=1} sen\,(i\pi x/L)\times \\
&\times\left[sen(k_i t)\frac{1}{k_i}\frac{2}{L}\int_0^L \psi(x')\,sen\,(i\pi x'/L)\,dx'\right. \\
&+cos(k_i t)\frac{2}{L}\int_0^L\left(\varphi(x')-\frac{u_2-u_1}{L}x'-u_1\right)\,sen\,(i\pi x'/L)\,dx' \\
&+\frac{1}{k_i}sen(k_i t)\int_0^t cos(k_i t')f_i(t')dt' \\
&\left.-\frac{1}{k_i}cos(k_i t)\int_0^t sen(k_i t')f_i(t')dt'\right]\,,\quad k_i=vi\pi/L\,.
\end{aligned}
$$

Podemos verificar que $u(0,t)=u_1$ e $u(L,t)=u_2$, como esperado. Em $t=0$,

$$
\begin{aligned}
u(x,0) &= \frac{u_2-u_1}{L}x+u_1 \\
&+\sum_{i=1} sen\,(i\pi x/L)\times \\
&\times\left[\frac{2}{L}\int_0^L\left(\varphi(x')-\frac{u_2-u_1}{L}x'-u_1\right)\,sen\,(i\pi x'/L)\,dx'\right]\,, \\
&= \frac{u_2-u_1}{L}x+u_1 \\
&+\varphi(x)-\frac{u_2-u_1}{L}x-u_1=\varphi(x)\,,
\end{aligned}
$$

como esperado. Calculando $\partial u/\partial t$,

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \sum_{i=1} sen\ (i\pi x/L) \times \\
&\times \Bigg[k_i cos(k_i t) \frac{1}{k_i} \frac{2}{L} \int_0^L \psi(x')\, sen\ (i\pi x'/L)\, dx' \\
&- k_i sen(k_i t) \frac{2}{L} \int_0^L \left(\varphi(x') - \frac{u_2 - u_1}{L} x' - u_1 \right) sen\ (i\pi x'/L)\, dx' \\
&+ cos(k_i t) \int_0^t cos(k_i t') f_i(t') dt' \\
&+ sen(k_i t) \int_0^t sen(k_i t') f_i(t') dt' \\
&+ \frac{1}{k_i} sen(k_i t) cos(k_i t) f_i(t) \\
&- \frac{1}{k_i} cos(k_i t) sen(k_i t) f_i(t) \Bigg] , \\
&= \sum_{i=1} sen\ (i\pi x/L) \times \\
&\times \Bigg[k_i cos(k_i t) \frac{1}{k_i} \frac{2}{L} \int_0^L \psi(x')\, sen\ (i\pi x'/L)\, dx' \\
&- k_i sen(k_i t) \frac{2}{L} \int_0^L \left(\varphi(x') - \frac{u_2 - u_1}{L} x' - u_1 \right) sen\ (i\pi x'/L)\, dx' \\
&+ cos(k_i t) \int_0^t cos(k_i t') f_i(t') dt' \\
&+ sen(k_i t) \int_0^t sen(k_i t') f_i(t') dt' \Bigg] .
\end{aligned}
$$

Em $t = 0$,

$$
\begin{aligned}
\frac{\partial u(x,0)}{\partial t} &= \sum_{i=1} sen\ (i\pi x/L) \times \\
&\times \left[\frac{2}{L} \int_0^L \psi(x')\, sen\ (i\pi x'/L)\, dx' \right] , \\
&= \psi(x) ,
\end{aligned}
$$

como esperado. Vamos verificar agora que u *satisfaz a equação diferencial. Temos,*

$$
\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= \sum_{i=1} sen\,(i\pi x/L) \times \\
&\times \left[-k_i sen(k_i t) \frac{2}{L} \int_0^L \psi(x')\, sen\,(i\pi x'/L)\, dx' \right. \\
&-k_i^2 cos(k_i t) \frac{2}{L} \int_0^L \left(\varphi(x') - \frac{u_2 - u_1}{L} x' - u_1 \right) sen\,(i\pi x'/L)\, dx' \\
&-k_i sen(k_i t) \int_0^t cos(k_i t') f_i(t') dt' \\
&+k_i cos(k_i t) \int_0^t sen(k_i t') f_i(t') dt' \\
&+cos(k_i t) cos(k_i t) f_i(t) \\
&+sen(k_i t) sen(k_i t) f_i(t)] \;, \\
&= \sum_{i=1} sen\,(i\pi x/L) \times \\
&\times \left[-k_i sen(k_i t) \frac{2}{L} \int_0^L \psi(x')\, sen\,(i\pi x'/L)\, dx' \right. \\
&-k_i^2 cos(k_i t) \frac{2}{L} \int_0^L \left(\varphi(x') - \frac{u_2 - u_1}{L} x' - u_1 \right) sen\,(i\pi x'/L)\, dx' \\
&-k_i sen(k_i t) \int_0^t cos(k_i t') f_i(t') dt' \\
&+k_i cos(k_i t) \int_0^t sen(k_i t') f_i(t') dt' \\
&+f_i(t)] \;.
\end{aligned}
$$

Calculando as derivadas em relação a x,

$$
\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{u_2 - u_1}{L} \\
&+ \sum_{i=1} (i\pi/L) cos\,(i\pi x/L) \times \\
&\times \left[sen(k_i t) \frac{1}{k_i} \frac{2}{L} \int_0^L \psi(x')\, sen\,(i\pi x'/L)\, dx' \right. \\
&+ cos(k_i t) \frac{2}{L} \int_0^L \left(\varphi(x') - \frac{u_2 - u_1}{L} x' - u_1 \right) sen\,(i\pi x'/L)\, dx' \\
&+ \frac{1}{k_i} sen(k_i t) \int_0^t cos(k_i t') f_i(t') dt' \\
&\left. - \frac{1}{k_i} cos(k_i t) \int_0^t sen(k_i t') f_i(t') dt' \right], \\
\frac{\partial^2 u}{\partial x^2} &= -\sum_{i=1} (i\pi/L)^2 sen\,(i\pi x/L) \times \\
&\times \left[sen(k_i t) \frac{1}{k_i} \frac{2}{L} \int_0^L \psi(x')\, sen\,(i\pi x'/L)\, dx' \right. \\
&+ cos(k_i t) \frac{2}{L} \int_0^L \left(\varphi(x') - \frac{u_2 - u_1}{L} x' - u_1 \right) sen\,(i\pi x'/L)\, dx' \\
&+ \frac{1}{k_i} sen(k_i t) \int_0^t cos(k_i t') f_i(t') dt' \\
&\left. - \frac{1}{k_i} cos(k_i t) \int_0^t sen(k_i t') f_i(t') dt' \right].
\end{aligned}
$$

Assim,

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} - v^2 \frac{\partial^2 u}{\partial x^2} = \\
&= \sum_{i=1} sen\,(i\pi x/L) \times \\
&\times \left[-k_i sen(k_i t) \frac{2}{L} \int_0^L \psi(x')\, sen\,(i\pi x'/L)\, dx' \right. \\
&-k_i^2 cos(k_i t) \frac{2}{L} \int_0^L \left(\varphi(x') - \frac{u_2 - u_1}{L} x' - u_1 \right) sen\,(i\pi x'/L)\, dx' \\
&-k_i sen(k_i t) \int_0^t cos(k_i t') f_i(t') dt' \\
&+k_i cos(k_i t) \int_0^t sen(k_i t') f_i(t') dt' \\
&+ f_i(t)] \\
&+v^2 \sum_{i=1} (i\pi/L)^2 sen\,(i\pi x/L) \times \\
&\times \left[sen(k_i t) \frac{1}{k_i} \frac{2}{L} \int_0^L \psi(x')\, sen\,(i\pi x'/L)\, dx' \right. \\
&+cos(k_i t) \frac{2}{L} \int_0^L \left(\varphi(x') - \frac{u_2 - u_1}{L} x' - u_1 \right) sen\,(i\pi x'/L)\, dx' \\
&+\frac{1}{k_i} sen(k_i t) \int_0^t cos(k_i t') f_i(t') dt' \\
&\left. -\frac{1}{k_i} cos(k_i t) \int_0^t sen(k_i t') f_i(t') dt' \right], \\
= \quad & \sum_{i=1} sen\,(i\pi x/L) f_i(t) = f(x,t)\,,
\end{aligned}
$$

como esperado.

10. Considere o problema 5 com as condições gerais,

$$
\begin{aligned}
u(0,t) = \mu_1(t)\,, \quad u(L,t) = \mu_2(t)\,, \\
u(x,0) = \varphi(x)\,, \quad u_t(x,0) = \psi(x)\,.
\end{aligned}
$$

Escrevemos a solução como (Tijonov [12], p. 120),

$$
u(x,t) = U(x,t) + V(x,t)\,,
$$

com,

$$U(x,t) = \frac{\mu_2(t) - \mu_1(t)}{L}x + \mu_1(t)\,,$$

A função U satisfaz as condições,

$$\begin{aligned}
&U(0,t) = \mu_1(t)\,,\\
&U(L,t) = \mu_2(t)\,,\\
&U(x,0) = \frac{\mu_2(0) - \mu_1(0)}{L}x + \mu_1(0)\,,\\
&U_t(x,0) = \frac{\mu_2'(0) - \mu_1'(0)}{L}x + \mu_1'(0)\,.
\end{aligned}$$

As condições de contorno para u ficam,

$$\begin{aligned}
&u(0,t) = \mu_1(t) + V(0,t) = \mu_1(t)\,,\\
&u(L,t) = \mu_2(t) + V(L,t) = \mu_2(t)\,,\\
&u(x,0) = \frac{\mu_2(0) - \mu_1(0)}{L}x + \mu_1(0) + V(x,0) = \varphi(x)\,,\\
&u_t(x,0) = \frac{\mu_2'(0) - \mu_1'(0)}{L}x + \mu_1'(0) + V_t(x,0) = \psi(x)\,.
\end{aligned}$$

Portanto, para V temos,

$$\begin{aligned}
&V(0,t) = 0\,,\\
&V(L,t) = 0\,,\\
&V(x,0) = \varphi(x) - \frac{\mu_2(0) - \mu_1(0)}{L}x - \mu_1(0)\,,\\
&V_t(x,0) = \psi(x) - \frac{\mu_2'(0) - \mu_1'(0)}{L}x - \mu_1'(0)\,.
\end{aligned}$$

Substituindo u na equação diferencial,

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} - v^2\frac{\partial^2 u}{\partial x^2} = f(x,t)\,,\\
&\frac{\mu_2''(t) - \mu_1''(t)}{L}x + \mu_1''(t) + \frac{\partial^2 V}{\partial t^2} - v^2\frac{\partial^2 V}{\partial x^2} = f(x,t)\,.
\end{aligned}$$

Portanto V satisfaz a equação,

$$\frac{\partial^2 V}{\partial t^2} - v^2 \frac{\partial^2 V}{\partial x^2} = f(x,t) - \frac{\mu_2''(t) - \mu_1''(t)}{L} x - \mu_1''(t)\,,$$

com as condições acima. Esse problema foi resolvido acima (probl. 7), logo podemos escrever V *como,*

$$\begin{aligned} V(x,t) &= \sum_{i=1} sen\,(i\pi x/L) \times \\ &\times \left[sen(k_i t)\frac{1}{k_i}\frac{2}{L}\int_0^L \left(\psi(x') - \frac{\mu_2'(0) - \mu_1'(0)}{L} x' - \mu_1'(0) \right) sen\,(i\pi x'/L)\,dx' \right. \\ &+ cos(k_i t)\frac{2}{L}\int_0^L \left(\varphi(x') - \frac{\mu_2(0) - \mu_1(0)}{L} x' - \mu_1(0) \right) sen\,(i\pi x'/L)\,dx' \\ &+ \frac{1}{k_i} sen(k_i t)\int_0^t cos(k_i t') f_i(t') dt' \\ &\left. - \frac{1}{k_i} cos(k_i t)\int_0^t sen(k_i t') f_i(t') dt' \right]\,,\quad k_i = v(i\pi/L)\,. \end{aligned}$$

Lembramos que,

$$f(x,t) - \frac{\mu_2''(t) - \mu_1''(t)}{L} x - \mu_1''(t) = \sum_{i=1} f_i(t)\, sen\,(i\pi x/L)\,,$$

com,

$$f_i(t) = \frac{2}{L}\int_0^L \left(f(x',t) - \frac{\mu_2''(t) - \mu_1''(t)}{L} x' - \mu_1''(t) \right) sen\,(i\pi x'/L) dx'\,.$$

A solução u é portanto,

$$
\begin{aligned}
u(x,t) &= U(x,t) + V(x,t)\,,\\
&= \frac{\mu_2(t)-\mu_1(t)}{L}x + \mu_1(t)\\
&\quad + \sum_{i=1} sen\,(i\pi x/L)\times\\
&\quad \times\Bigg[sen(k_i t)\frac{1}{k_i}\frac{2}{L}\int_0^L\left(\psi(x') - \frac{\mu_2'(0)-\mu_1'(0)}{L}x' - \mu_1'(0)\right)\,sen\,(i\pi x'/L)\,dx'\\
&\quad + cos(k_i t)\frac{2}{L}\int_0^L\left(\varphi(x') - \frac{\mu_2(0)-\mu_1(0)}{L}x' - \mu_1(0)\right)\,sen\,(i\pi x'/L)\,dx'\\
&\quad + \frac{1}{k_i}sen(k_i t)\int_0^t cos(k_i t') f_i(t') dt'\\
&\quad - \frac{1}{k_i}cos(k_i t)\int_0^t sen(k_i t') f_i(t') dt'\Bigg]\,,\quad k_i = v(i\pi/L)\,.
\end{aligned}
$$

Vamos verificar que a solução acima satisfaz as condições iniciais, de contorno, e a equação diferencial. Temos,

$$
\begin{aligned}
u(0,t) &= \mu_1(t)\,,\\
u(L,t) &= \mu_2(t)\,,\\
u(x,0) &= \frac{\mu_2(0)-\mu_1(0)}{L}x + \mu_1(0)\\
&\quad + \sum_{i=1} sen\,(i\pi x/L)\times\\
&\quad \times\left[\frac{2}{L}\int_0^L\left(\varphi(x') - \frac{\mu_2(0)-\mu_1(0)}{L}x' - \mu_1(0)\right)\,sen\,(i\pi x'/L)\,dx'\right]\,,\\
&= \frac{\mu_2(0)-\mu_1(0)}{L}x + \mu_1(0)\\
&\quad + \varphi(x) - \frac{\mu_2(0)-\mu_1(0)}{L}x - \mu_1(0) = \varphi(x)\,,
\end{aligned}
$$

como esperado. Calculando agora $\partial u/\partial t$,

$$
\begin{aligned}
\frac{\partial u(x,t)}{\partial t} &= \frac{\mu_2'(t)-\mu_1'(t)}{L}x+\mu_1'(t) \\
&+\sum_{i=1} sen\ (i\pi x/L)\times \\
&\times\left[k_i cos(k_i t)\frac{1}{k_i}\frac{2}{L}\int_0^L\left(\psi(x')-\frac{\mu_2'(0)-\mu_1'(0)}{L}x'-\mu_1'(0)\right)\ sen\ (i\pi x'/L)\,dx'\right. \\
&-k_i sen(k_i t)\frac{2}{L}\int_0^L\left(\varphi(x')-\frac{\mu_2(0)-\mu_1(0)}{L}x'-\mu_1(0)\right)\ sen\ (i\pi x'/L)\,dx' \\
&+cos(k_i t)\int_0^t cos(k_i t')f_i(t')dt' \\
&\left.+sen(k_i t)\int_0^t sen(k_i t')f_i(t')dt'\right]\ , \\
&+\frac{1}{k_i}sen(k_i t)cos(k_i t)f_i(t) \\
&\left.-\frac{1}{k_i}cos(k_i t)sen(k_i t)f_i(t)\right]\ , \\
&= \frac{\mu_2'(t)-\mu_1'(t)}{L}x+\mu_1'(t) \\
&+\sum_{i=1} sen\ (i\pi x/L)\times \\
&\times\left[k_i cos(k_i t)\frac{1}{k_i}\frac{2}{L}\int_0^L\left(\psi(x')-\frac{\mu_2'(0)-\mu_1'(0)}{L}x'-\mu_1'(0)\right)\ sen\ (i\pi x'/L)\,dx'\right. \\
&-k_i sen(k_i t)\frac{2}{L}\int_0^L\left(\varphi(x')-\frac{\mu_2(0)-\mu_1(0)}{L}x'-\mu_1(0)\right)\ sen\ (i\pi x'/L)\,dx' \\
&+cos(k_i t)\int_0^t cos(k_i t')f_i(t')dt' \\
&\left.+sen(k_i t)\int_0^t sen(k_i t')f_i(t')dt'\right]\ .
\end{aligned}
$$

$Em\ t=0,$

$$
\begin{aligned}
\frac{\partial u(x,0)}{\partial t} &= \frac{\mu_2'(0)-\mu_1'(0)}{L}x+\mu_1'(0) \\
&\quad +\sum_{i=1} sen\,(i\pi x/L)\times \\
&\quad \times\left[\frac{2}{L}\int_0^L\left(\psi(x')-\frac{\mu_2'(0)-\mu_1'(0)}{L}x'-\mu_1'(0)\right)\,sen\,(i\pi x'/L)\,dx'\right], \\
&= \frac{\mu_2'(0)-\mu_1'(0)}{L}x+\mu_1'(0) \\
&\quad +\psi(x)-\frac{\mu_2'(0)-\mu_1'(0)}{L}x-\mu_1'(0)=\psi(x)\,,
\end{aligned}
$$

como esperado. Calculando as demais derivadas de u,

$$
\begin{aligned}
\frac{\partial^2 u(x,t)}{\partial t^2} &= \frac{\mu_2''(t)-\mu_1''(t)}{L}x+\mu_1''(t) \\
&+\sum_{i=1} sen\,(i\pi x/L)\times \\
&\times\left[-k_i^2 sen(k_i t)\frac{1}{k_i}\frac{2}{L}\int_0^L\left(\psi(x')-\frac{\mu_2'(0)-\mu_1'(0)}{L}x'-\mu_1'(0)\right)\,sen\,(i\pi x'/L)\,dx'\right. \\
&-k_i^2 cos(k_i t)\frac{2}{L}\int_0^L\left(\varphi(x')-\frac{\mu_2(0)-\mu_1(0)}{L}x'-\mu_1(0)\right)\,sen\,(i\pi x'/L)\,dx' \\
&-k_i sen(k_i t)\int_0^t cos(k_i t')f_i(t')dt' \\
&+k_i cos(k_i t)\int_0^t sen(k_i t')f_i(t')dt' \\
&+cos(k_i t)cos(k_i t)f_i(t) \\
&\left.+sen(k_i t)sen(k_i t)f_i(t)\right]\,, \\
&= \frac{\mu_2''(t)-\mu_1''(t)}{L}x+\mu_1''(t) \\
&+\sum_{i=1} sen\,(i\pi x/L)\times \\
&\times\left[-k_i^2 sen(k_i t)\frac{1}{k_i}\frac{2}{L}\int_0^L\left(\psi(x')-\frac{\mu_2'(0)-\mu_1'(0)}{L}x'-\mu_1'(0)\right)\,sen\,(i\pi x'/L)\,dx'\right. \\
&-k_i^2 cos(k_i t)\frac{2}{L}\int_0^L\left(\varphi(x')-\frac{\mu_2(0)-\mu_1(0)}{L}x'-\mu_1(0)\right)\,sen\,(i\pi x'/L)\,dx' \\
&-k_i sen(k_i t)\int_0^t cos(k_i t')f_i(t')dt' \\
&+k_i cos(k_i t)\int_0^t sen(k_i t')f_i(t')dt' \\
&\left.+f_i(t)\right]\,.
\end{aligned}
$$

As derivadas em relação a x são,

$$
\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\mu_2(t)-\mu_1(t)}{L} \\
&+\sum_{i=1}(i\pi/L)cos\,(i\pi x/L)\times \\
&\times\left[sen(k_i t)\frac{1}{k_i}\frac{2}{L}\int_0^L\left(\psi(x')-\frac{\mu_2'(0)-\mu_1'(0)}{L}x'-\mu_1'(0)\right)\,sen\,(i\pi x'/L)\,dx'\right. \\
&+cos(k_i t)\frac{2}{L}\int_0^L\left(\varphi(x')-\frac{\mu_2(0)-\mu_1(0)}{L}x'-\mu_1(0)\right)\,sen\,(i\pi x'/L)\,dx' \\
&+\frac{1}{k_i}sen(k_i t)\int_0^t cos(k_i t')f_i(t')dt' \\
&\left.-\frac{1}{k_i}cos(k_i t)\int_0^t sen(k_i t')f_i(t')dt'\right]\,, \\
\frac{\partial^2 u}{\partial x^2} &= -\sum_{i=1}(i\pi/L)^2 sen\,(i\pi x/L)\times \\
&\times\left[sen(k_i t)\frac{1}{k_i}\frac{2}{L}\int_0^L\left(\psi(x')-\frac{\mu_2'(0)-\mu_1'(0)}{L}x'-\mu_1'(0)\right)\,sen\,(i\pi x'/L)\,dx'\right. \\
&+cos(k_i t)\frac{2}{L}\int_0^L\left(\varphi(x')-\frac{\mu_2(0)-\mu_1(0)}{L}x'-\mu_1(0)\right)\,sen\,(i\pi x'/L)\,dx' \\
&+\frac{1}{k_i}sen(k_i t)\int_0^t cos(k_i t')f_i(t')dt' \\
&\left.-\frac{1}{k_i}cos(k_i t)\int_0^t sen(k_i t')f_i(t')dt'\right]\,.
\end{aligned}
$$

Substituindo na equação diferencial,

$$
\begin{aligned}
& \frac{\partial^2 u}{\partial t^2} - v^2 \frac{\partial^2 u}{\partial x^2} = \\
& = \frac{\mu_2''(t) - \mu_1''(t)}{L} x + \mu_1''(t) \\
& + \sum_{i=1} sen\,(i\pi x/L) \times \\
& \times \left[-k_i^2 sen(k_i t) \frac{1}{k_i} \frac{2}{L} \int_0^L \left(\psi(x') - \frac{\mu_2'(0) - \mu_1'(0)}{L} x' - \mu_1'(0) \right) sen\,(i\pi x'/L)\, dx' \right. \\
& -k_i^2 cos(k_i t) \frac{2}{L} \int_0^L \left(\varphi(x') - \frac{\mu_2(0) - \mu_1(0)}{L} x' - \mu_1(0) \right) sen\,(i\pi x'/L)\, dx' \\
& -k_i sen(k_i t) \int_0^t cos(k_i t') f_i(t') dt' \\
& +k_i cos(k_i t) \int_0^t sen(k_i t') f_i(t') dt' \\
& \left. + f_i(t) \right] \\
& +v^2 \sum_{i=1} (i\pi/L)^2 sen\,(i\pi x/L) \times \\
& \times \left[sen(k_i t) \frac{1}{k_i} \frac{2}{L} \int_0^L \left(\psi(x') - \frac{\mu_2'(0) - \mu_1'(0)}{L} x' - \mu_1'(0) \right) sen\,(i\pi x'/L)\, dx' \right. \\
& +cos(k_i t) \frac{2}{L} \int_0^L \left(\varphi(x') - \frac{\mu_2(0) - \mu_1(0)}{L} x' - \mu_1(0) \right) sen\,(i\pi x'/L)\, dx' \\
& + \frac{1}{k_i} sen(k_i t) \int_0^t cos(k_i t') f_i(t') dt' \\
& \left. - \frac{1}{k_i} cos(k_i t) \int_0^t sen(k_i t') f_i(t') dt' \right] \\
= \; & \frac{\mu_2''(t) - \mu_1''(t)}{L} x + \mu_1''(t) + \sum_{i=1} f_i(t) sen\,(i\pi x/L)\,, \\
= \; & \frac{\mu_2''(t) - \mu_1''(t)}{L} x + \mu_1''(t) + f(x,t) - \frac{\mu_2''(t) - \mu_1''(t)}{L} x - \mu_1''(t)\,, \\
= \; & f(x,t)\,,
\end{aligned}
$$

como esperado. Lembramos que,

$$
f(x,t) - \frac{\mu_2''(t) - \mu_1''(t)}{L} x - \mu_1''(t) = \sum_{i=1} f_i(t)\, sen\,(i\pi x/L)\,,
$$

com,

$$f_i(t) = \frac{2}{L}\int_0^L \left(f(x',t) - \frac{\mu_2''(t) - \mu_1''(t)}{L}x' - \mu_1''(t)\right) sen\ (i\pi x'/L)dx' \,.$$

Outra forma de resolver o problema é escrever a solução na forma,

$$u(x,t) = u_1(x,t) + u_2(x,t) + u_3(x,t) + u_4(x,t)\,,$$

com as condições,

$$\begin{array}{l} u_1(0,t) = 0\,,\;\; u_2(0,t) = \mu_1(t)\,,\;\; u_3(0,t) = 0\,,\;\; u_4(0,t) = 0\,, \\ u_1(L,t) = 0\,,\;\; u_2(L,t) = 0\,,\;\; u_3(L,t) = \mu_2(t)\,,\;\; u_4(L,t) = 0\,, \\ u_1(x,0) = \varphi(x)\,,\;\; u_2(x,0) = 0\,,\;\; u_3(x,0) = 0\,,\;\; u_4(x,0) = 0\,, \\ u_{1t}(x,0) = \psi(x)\,,\;\; u_{2t}(x,0) = 0\,,\;\; u_{3t}(x,0) = 0\,,\;\; u_{4t}(x,0) = 0\,. \end{array}$$

11. Determine a solução $u(x,t)$ da equação da onda na região aberta,

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2}\,,$$

$$0 < x < L\,,\;\; 0 < t\,.$$

As condições inicial e de fronteira são,

$$\begin{array}{l} u(0,t) = \mu_1(t)\,,\;\; u(L,t) = \mu_2(t)\,, \\ u(x,0) = \varphi(x)\,,\;\; u_t(x,0) = \psi(x)\,, \end{array}$$

que satisfazem as condições de conjunção,

$$\begin{array}{l} \varphi(0) = \mu_1(0) = u(0,0)\,, \\ \varphi(L) = \mu_2(0) = u(L,0)\,. \end{array}$$

Usando o resultado do problema anterior, com $f(x,t) = 0$, temos,

$$
\begin{aligned}
u(x,t) &= \frac{\mu_2(t)-\mu_1(t)}{L}x+\mu_1(t) \\
&+\sum_{i=1} sen\ (i\pi x/L)\times \\
&\times\left[sen(k_it)\frac{1}{k_i}\frac{2}{L}\int_0^L\left(\psi(x')-\frac{\mu_2'(0)-\mu_1'(0)}{L}x'-\mu_1'(0)\right)\ sen\ (i\pi x'/L)\,dx'\right. \\
&+cos(k_it)\frac{2}{L}\int_0^L\left(\varphi(x')-\frac{\mu_2(0)-\mu_1(0)}{L}x'-\mu_1(0)\right)\ sen\ (i\pi x'/L)\,dx' \\
&+\frac{1}{k_i}sen(k_it)\int_0^t cos(k_it')f_i(t')dt' \\
&\left.-\frac{1}{k_i}cos(k_it)\int_0^t sen(k_it')f_i(t')dt'\right]\ ,\quad k_i=v(i\pi/L)\,.
\end{aligned}
$$

Temos agora,

$$
-\frac{\mu_2''(t)-\mu_1''(t)}{L}x-\mu_1''(t)=\sum_{i=1} f_i(t)\,sen\ (i\pi x/L)\,,
$$

com,

$$
f_i(t)=\frac{2}{L}\int_0^L\left(-\frac{\mu_2''(t)-\mu_1''(t)}{L}x'-\mu_1''(t)\right)\ sen\ (i\pi x'/L)dx'\,.
$$

12. Resolva o problema (Spiegel [7], probl. 1.24),

$$
\frac{\partial^2 u}{\partial t^2}=16\frac{\partial^2 u}{\partial x^2}\,,\quad 0<x<2\,,\quad t>0\,,
$$

com as condições,

$$
u(0,t)=u(2,t)=0\,,\quad u(x,0)=6\,\mathrm{sen}\,\pi x-3\,\mathrm{sen}\,4\pi x\,,\quad y_t(x,0)=0\,,\quad |u(x,t)|<M\,.
$$

13. Resolva o problema anterior com $y(x,0)=f(x)$, expandindo $f(x)$ em uma série de senos (Spiegel [7], probl. 1.46).

14. Uma corda de comprimento L está sobre o eixo x. Em $t=0$ a forma da corda é $f(x)$ e a velocidade inicial é zero. Calcule $u(x,t)$ (Spiegel [7], probl. 2.32).

15. Uma corda possui extremidades em $x=0$ e $x=2$. Sendo $u(x,0)=f(x)=0,03x(2-x)$ e a velocidade inicial zero, calcule $u(x,t)$.

16. Resolva o problema (Spiegel [7], probl. 2.71),

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} - g\,, \quad 0 < x < L\,, \quad t > 0\,,$$

com,

$$u(0,t) = u(L,t) = 0\,, \quad u(x,0) = f(x)\,, \quad u_t(x,0) = 0\,, \quad |u(x,t| < M\,.$$

17. Considere o problema anterior com $u_t(x,0) = g(x)$.

18. Uma corda infinita possui deslocamento inicial $u(x,0) = f(x)$. Calcule $u(x,t)$ desprezando a gravidade.

19. Uma corda infinita possui deslocamento inicial $u(x,0) = f(x)$ e $u_t(x,0) = g(x)$. Calcule $u(x,t)$ desprezando a gravidade (Spiegel [7], probl. 5.50).

20. Considere o problema anterior com a aceleração da gravidade (Spiegel [7], probl. 5.51).

21. Mostre que o problema de valores de contorno,

$$g(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left[\tau(x)\frac{\partial u}{\partial x}\right] + h(x)u\,,$$
$$u(0,t) = u(L,t) = 0\,, \quad u(x,0) = f(x)\,, \quad u_t(x,0) = 0\,, \quad |u(t,x)| < M\,,$$

é um problema de Sturm-Liouville. Calcule $u(x,t)$ (Spiegel [7], probl. 3.38).

22. Considere o problema anterior se as condições de contorno são $u_x(0,t) = h_1 u(0,t)$, $u_x(L,t) = h_2 u(L,t)$ (Spiegel [7], probl. 3.39).

23. Resolva o problema de valores de contorno (Spiegel [7], probl. 3.41),

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}\,,$$
$$u(0,t) = u_x(L,t) = 0\,, \quad u(x,0) = f(x)\,, \quad u_t(x,0) = 0\,, \quad |u(x,t)| < M\,.$$

24. Mostre que $y(x,t) = F(2x+5t) + G(2x-5t)$ é uma solução geral da equação (Spiegel [7], probl. 1.14),

$$4\frac{\partial^2 y}{\partial t^2} = 25\frac{\partial^2 y}{\partial x^2}\,.$$

Encontre uma solução particular satisfazendo as condições,

$$y(0,t) = y(\pi,t) = 0\,, \quad y(x,0) = \text{ sen } 2x\,, \quad y_t(x,0) = 0\,.$$

25. Uma corda em $0 < x < L$ é deslocada uma distância h no seu ponto médio, e então é solta. Calcule $y(x,t)$ (Spiegel [7], probl. 1.31).

26. Calcule a solução do problema,

$$\frac{\partial^2 y}{\partial t^2} = 4\frac{\partial^2 y}{\partial x^2}\,,$$

com as condições,

$$y(0,t) = y(5,t) = 0\,,\ \ y(x,0) = 0\,,\ \ y_t(x,0) = f(x)\,,$$

se (a) $f(x) = 5\,\mathrm{sen}\ \pi x$, (b) $f(x) = 3\,\mathrm{sen}\ 2\pi x - 2\,\mathrm{sen}\ 5\pi x$ (Spiegel [7], probl. 1.44).

27. Calcule $u(x,t)$ em $-\infty < x < +\infty$, $t > 0$ com,

$$u(x,0) = h(x)\,,\ \ \frac{\partial u(x,0)}{\partial t} = w(x)\,.$$

Escrevemos a solução como,

$$\begin{aligned} u(x,t) &= \int_0^\infty dk[a_1\cos(kx) + a_2\,sen\,(kx)]\times \\ &\quad \times[b_1\cos(kvt) + b_2\,sen\ (kvt)]\,. \end{aligned}$$

A condição de contorno nos dá,

$$u(x,0) = h(x) = \int_0^\infty dk[a_1\cos(kx) + a_2\,sen\,(kx)]b_1\,,$$

portanto,

$$a_1 b_1 = \frac{1}{\pi}\int_{-\infty}^\infty h(\xi)\cos k\xi d\xi\,,$$
$$a_2 b_1 = \frac{1}{\pi}\int_{-\infty}^\infty h(\xi)\,sen\,k\xi d\xi\,.$$

Usando agora a condição de contorno para a velocidade inicial,

$$\frac{\partial u(x,0)}{\partial t} = w(x) = \int_0^\infty dk[a_1\cos(kx) + a_2\,sen\,(kx)]b_2 kv\,.$$

Portanto,

$$a_1 b_2 = \frac{1}{kv}\frac{1}{\pi}\int_{-\infty}^{\infty} w(\xi)\cos k\xi d\xi\,,$$
$$a_2 b_2 = \frac{1}{kv}\frac{1}{\pi}\int_{-\infty}^{\infty} w(\xi)\,sen\,k\xi d\xi\,.$$

A solução é então,

$$\begin{aligned} u(x,t) &= \int_0^\infty dk\cos(kx)\cos(kvt)\frac{1}{\pi}\int_{-\infty}^{\infty} h(\xi)\cos k\xi d\xi \\ &+ \int_0^\infty dk\cos(kx)\,sen\,(kvt)\frac{1}{kv}\frac{1}{\pi}\int_{-\infty}^{\infty} w(\xi)\cos k\xi d\xi \\ &+ \int_0^\infty dk\,sen\,(kx)]\cos(kvt)\frac{1}{\pi}\int_{-\infty}^{\infty} h(\xi)\,sen\,k\xi d\xi \\ &+ \int_0^\infty dk\,sen\,(kx)]\,sen\,(kvt)\frac{1}{kv}\frac{1}{\pi}\int_{-\infty}^{\infty} w(\xi)\,sen\,k\xi d\xi\,. \end{aligned}$$

28. Calcule $u(x,t)$ em $0 < x < +\infty$, $t > 0$ com,

$$u(0,t) = f(t)\,,\quad \frac{\partial u(0,t)}{\partial x} = g(t)\,.$$

O intervalo para t não é infinito, que é o intervalo em que precisaremos expandir as funções f(t) e g(t). Escrevemos a então a solução como,

$$u(x,t) = \int_0^\infty dk[a_1\cos(kx) + a_2\,sen\,(kx)]\,sen\,(kvt)\,.$$

As condições de contorno nos dão,

$$\begin{aligned} u(0,t) &= f(t) = \int_0^\infty dk a_1\,sen\,(kvt) = \frac{1}{v}\int_0^\infty d(kv)a_1\,sen\,(kvt)\,, \\ &= \frac{2}{\pi}\frac{\pi}{2v}\int_0^\infty dk' a_1\,sen\,(k't)\,, \\ \frac{\partial u(0,t)}{\partial x} &= g(t) = \int_0^\infty dk a_2 k\,sen\,(kvt) = \frac{1}{v^2}\int_0^\infty d(kv)a_2(kv)\,sen\,(kvt)\,, \\ &= \frac{2}{\pi}\frac{\pi}{2v^2}\int_0^\infty dk' a_2 k'\,sen\,(k't)\,. \end{aligned}$$

Portanto,

$$\frac{\pi}{2v}a_1 = \int_0^\infty f(\xi)\, sen\, k'\xi d\xi\,,$$
$$\frac{\pi}{2v^2}a_2 k' = \int_0^\infty g(\xi)\, sen\, k'\xi d\xi\,,$$

ou,

$$a_1 = \frac{2v}{\pi}\int_0^\infty f(\xi)\, sen\, k'\xi d\xi\,,$$
$$a_2 = \frac{2v^2}{k'\pi}\int_0^\infty g(\xi)\, sen\, k'\xi d\xi\,.$$

A solução é então,

$$\begin{aligned} u(x,t) &= \int_0^\infty dk \cos(kx)\, sen\ (kvt)\frac{2v}{\pi}\int_0^\infty f(\xi)\, sen\,(kv\xi) d\xi \\ &+ \int_0^\infty dk\, sen\,(kx)\, sen\ (kvt)\frac{2v}{k\pi}\int_0^\infty g(\xi)\, sen\,(kv\xi) d\xi\,. \end{aligned}$$

29. Calcule $u(x,t)$ em $0 < x < +\infty$, $t > 0$ com,

$$u(x,0) = h(x)\,, \quad \frac{\partial u(x,0)}{\partial t} = w(x)\,.$$

Escrevemos a solução como,

$$u(x,t) = \int_0^\infty dk\, sen\,(kx)[b_1\cos(kvt) + b_2\, sen\ (kvt)]\,.$$

As condições de contorno nos dão,

$$\begin{aligned} u(x,0) &= h(x) = \int_0^\infty dk\, sen\,(kx) b_1 = \frac{2}{\pi}\int_0^\infty dk\, sen\,(kx)\frac{\pi}{2}b_1\,, \\ \frac{\partial u(x,0)}{\partial t} &= w(x) = \int_0^\infty dk\, sen\,(kx) b_2 kv = \frac{2}{\pi}\int_0^\infty dk\, sen\,(kx)\frac{\pi}{2}b_2 kv\,, \end{aligned}$$

logo,

$$
\begin{aligned}
\frac{\pi}{2} b_1 &= \int_0^\infty h(\xi)\, sen\,(k\xi) d\xi \,, \\
\frac{\pi}{2} b_2 k v &= \int_0^\infty \int_0^\infty w(\xi)\, sen\,(k\xi) d\xi \,,
\end{aligned}
$$

ou,

$$
\begin{aligned}
b_1 &= \frac{2}{\pi} \int_0^\infty h(\xi)\, sen\,(k\xi) d\xi \,, \\
b_2 &= \frac{2}{\pi k v} \int_0^\infty \int_0^\infty w(\xi)\, sen\,(k\xi) d\xi \,.
\end{aligned}
$$

A solução é então,

$$
\begin{aligned}
u(x,t) \;=\; & \int_0^\infty dk\, sen\,(kx) \cos(kvt) \frac{2}{\pi} \int_0^\infty h(\xi)\, sen\,(k\xi) d\xi \\
& + \int_0^\infty dk\, sen\,(kx)\, sen\,(kvt) \frac{2}{\pi k v} \int_0^\infty \int_0^\infty w(\xi)\, sen\,(k\xi) d\xi \,.
\end{aligned}
$$

30. Considere a equação diferencial,

$$
\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2} + \alpha^2 u \,,
$$

no intervalo $0 < x < L$, $t > 0$, com condições de contorno não-homogêneas,

$$
u(0,t) = \mu_1(t) \,, \;\; u(L,t) = \mu_2(t) \,,
$$

e condições iniciais,

$$
u(x,0) = \varphi(x) \,, \;\; u_t(x,0) = g(x) \,.
$$

31. Encontre a solução da equação diferencial,

$$
\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2} \,,
$$

no intervalo $0 < x < L$, $t > 0$, com condições de contorno,

$$
\begin{aligned}
u_x(0,t) &= h[u(0,t) - \theta(t)] \,, \\
u_x(L,t) &= -h[u(L,t) - \theta(t)] \,,
\end{aligned}
$$

e condições iniciais,

$$
u(x,0) = \varphi(x) \,, \;\; u_t(x,0) = g(x) \,.
$$

Considerando $F(x,y)$

A equação (5) fica agora,

$$\frac{d^2F}{dx^2}+\frac{d^2F}{dy^2}+k^2F=0\,. \tag{11}$$

Substituindo $F(x,y)=X(x)Y(y)$,

$$\begin{aligned}&X''Y+XY''+k^2XY=0\,,\\&\frac{X''}{X}+\frac{Y''}{Y}+k^2=0\,,\end{aligned}$$

ou,

$$\frac{X''}{X}=-\frac{Y''}{Y}-k^2=-\alpha^2\,.$$

Obtemos assim as seguintes equações para X e Y,

$$\begin{aligned}&X''+\alpha^2X=0\,,\\&Y''+(k^2-\alpha^2)Y=0\,.\end{aligned}$$

As soluções das equações acima são,

$$\begin{aligned}&X(x)=a_1\cos(\alpha x)+a_2\,\mathrm{sen}\,(\alpha x)\,,\\&Y(y)=\begin{cases}b_1\cos(\mu y)+b_2\,\mathrm{sen}\,(\mu y)\,, & \mu=\sqrt{k^2-\alpha^2}\,,\ \ k^2>\alpha^2\,,\\ c_1\cosh(\mu y)+c_2\,\mathrm{senh}\,(\mu y)\,, & \mu=\sqrt{\alpha^2-k^2}\,,\ \ k^2<\alpha^2\,.\end{cases}\end{aligned}$$

Também podemos escolher a constante de separação α como,

$$\frac{X''}{X}=-\frac{Y''}{Y}-k^2=+\alpha^2\,.$$

Obtemos agora,

$$\begin{aligned}&X''-\alpha^2X=0\,,\\&Y''+(k^2+\alpha^2)Y=0\,,\end{aligned}$$

com solução,

$$X(x) = a_1 \cosh(\alpha x) + a_2 \, \mathrm{senh}\,(\alpha x)\,,$$
$$Y(y) = b_1 \cos(\mu y) + b_2 \, \mathrm{sen}\,(\mu y)\,, \quad \mu^2 = k^2 + \alpha^2\,.$$

Se temos $\alpha = 0$,

$$\frac{X''}{X} = -\frac{Y''}{Y} - k^2 = 0\,,$$

e obtemos,

$$X'' = 0\,,$$
$$Y'' + k^2 Y = 0\,.$$

As soluções das equações acima são,

$$X(x) = a_1 x + a_2\,,$$
$$Y(y) = b_1 \cos(ky) + b_2 \, \mathrm{sen}\,(ky)\,, \quad \alpha = 0\,.$$

Também podemos ter para $\alpha = 0$,

$$X(x) = a_1 \cos(kx) + a_2 \, \mathrm{sen}\,(kx)\,,$$
$$Y(y) = b_1 y + b_2\,, \quad \alpha = 0\,.$$

A solução geral é então, usando o princípio de superposição,

$$
\begin{aligned}
u(x,y,t) &= \sum_{k,\alpha} [a_1 \cos(\alpha x) + a_2 \,\mathrm{sen}\,(\alpha x)][b_1 \cos(\mu y) + b_2 \,\mathrm{sen}\,(\mu y)] \times \\
&\quad \times [c_1 \cos(kvt) + c_2 \,\mathrm{sen}\,(kvt)] \,, \quad \mu = \sqrt{k^2 - \alpha^2} \,, \quad k^2 > \alpha^2 \,, \\
&= \sum_{k,\alpha} [a_1 \cos(\alpha x) + a_2 \,\mathrm{sen}\,(\alpha x)][b_1 \cosh(\mu y) + b_2 \,\mathrm{senh}\,(\mu y)] \times \\
&\quad \times [c_1 \cos(kvt) + c_2 \,\mathrm{sen}\,(kvt)] \,, \quad \mu = \sqrt{\alpha^2 - k^2} \,, \quad k^2 < \alpha^2 \,, \\
&= \sum_{k,\alpha} [a_1 \cosh(\alpha x) + a_2 \,\mathrm{senh}\,(\alpha x)][b_1 \cos(\mu y) + b_2 \,\mathrm{sen}\,(\mu y)] \times \\
&\quad \times [c_1 \cos(kvt) + c_2 \,\mathrm{sen}\,(kvt)] \,, \quad \mu^2 = k^2 + \alpha^2 \,, \\
&= \sum_{k} [a_1 x + a_2][b_1 \cos(ky) + b_2 \,\mathrm{sen}\,(ky)] \times \\
&\quad \times [c_1 \cos(kvt) + c_2 \,\mathrm{sen}\,(kvt)] \,, \quad \alpha = 0 \,, \\
&= \sum_{k} [a_1 \cos(kx) + a_2 \,\mathrm{sen}\,(kx)][b_1 y + b_2] \times \\
&\quad \times [c_1 \cos(kvt) + c_2 \,\mathrm{sen}\,(kvt)] \,, \quad \alpha = 0 \,. \qquad (12)
\end{aligned}
$$

2 Problemas

1. Uma membrana retangular em $0 < x < a$, $0 < y < b$ com lados fixos, possui posição inicial $u(x,y,0) = f(x,y)$ e velocidade inicial zero. Calcule $u(x,y,t)$ (Spiegel [7], probl. 2.33).

2. Calcule $u(x,y,t)$ em $0 < x < a$, $0 < y < b$, $t > 0$ com,

$$
\begin{aligned}
u(0,y,t) = 0 \,, \quad u(a,y,t) = 0 \,, \\
u(x,0,t) = 0 \,, \quad u(x,b,t) = 0 \,.
\end{aligned}
$$

A posição inicial é $h(x,y)$ velocidade inicial é $w(x,y)$ (Spiegel [7], probl. 2.61; probl. 2.60 com $a = b = 1$.

Escrevemos a solução como,

$$
\begin{aligned}
u(x,y,t) &= \sum_{k,\alpha} [a_1 \cos(\alpha x) + a_2 \,sen\,(\alpha x)][b_1 \cos(\mu y) + b_2 \,sen\,(\mu y)] \times \\
&\quad \times [c_1 \cos(kvt) + c_2 \,sen\,(kvt)] \,, \\
&\quad \mu = \sqrt{k^2 - \alpha^2} \,, \quad k^2 > \alpha^2 \,.
\end{aligned}
$$

As condições de contorno nos dão,

$$
\begin{aligned}
u(0,y,t) &= 0 = \sum_{k,\alpha} a_1[b_1\cos(\mu y) + b_2\, sen\,(\mu y)] \times \\
&\quad \times[c_1\cos(kvt) + c_2\, sen\;(kvt)]\,, \\
u(a,y,t) &= 0 = \sum_{k,\alpha}[a_1\cos(\alpha a) + a_2\, sen\,(\alpha a)][b_1\cos(\mu y) + b_2\, sen\,(\mu y)] \times \\
&\quad \times[c_1\cos(kvt) + c_2\, sen\;(kvt)]\,, \\
u(x,0,t) &= 0 = \sum_{k,\alpha}[a_1\cos(\alpha x) + a_2\, sen\,(\alpha x)]b_1 \times \\
&\quad \times[c_1\cos(kvt) + c_2\, sen\;(kvt)]\,, \\
u(x,b,t) &= 0 = \sum_{k,\alpha}[a_1\cos(\alpha x) + a_2\, sen\,(\alpha x)][b_1\cos(\mu b) + b_2\, sen\,(\mu b)] \times \\
&\quad \times[c_1\cos(kvt) + c_2\, sen\;(kvt)]\,, \\
&\quad \mu = \sqrt{k^2-\alpha^2}\,,\;\; k^2 > \alpha^2\,.
\end{aligned}
$$

Satisfazemos as condições acima escolhendo,

$$
\begin{aligned}
& a_1 = b_1 = 0\,, \\
& \alpha_i a = i\pi\,,\;\; i = 1,2,\ldots \\
& \mu_j b = j\pi\,,\;\; j = 1,2,\ldots
\end{aligned}
$$

A solução fica portanto,

$$
\begin{aligned}
u(x,y,t) &= \sum_{ij} a_2 b_2\, sen\,(i\pi x/a)\, sen\,(j\pi y/b) \times \\
&\quad \times[c_1\cos(k_{ij}vt) + c_2\, sen\;(k_{ij}vt)]\,, \\
&\quad \alpha_i = i\pi/a\,,\;\; \mu_j = j\pi/b\,,\;\; i,j = 1,2,\ldots \\
&\quad k_{ij}^2 = \mu_j^2 + \alpha_i^2\,,\;\; k^2 > \alpha^2\,.
\end{aligned}
$$

Fazendo agora $a_2 = b_2 = 1$*, ainda temos que determinar* c_1 *e* c_2*. Usando a posição inicial* $h(x,y)$ *e a velocidade inicial* $w(x,y)$*,*

$$
\begin{aligned}
u(x,y,0) &= h(x,y) = \sum_{ij} sen\,(i\pi x/a)\, sen\,(j\pi y/b) c_1\,, \\
\frac{\partial u(x,y,0)}{\partial t} &= w(x,y) = \sum_{ij} sen\,(i\pi x/a)\, sen\,(j\pi y/b) c_2 k_{ij} v
\end{aligned}
$$

As expressões acima são expansões em série de senos de Fourier, logo,

$$\sum_i c_1 \, sen\,(i\pi x/a) = \frac{2}{b}\int_0^b h(x,\eta)\, sen\,(j\pi\eta/b) d\eta\,,$$
$$\sum_i c_2 k_{ij} v \, sen\,(i\pi x/a) = \frac{2}{b}\int_0^b w(x,\eta)\, sen\,(j\pi\eta/b) d\eta\,.$$

Continuando, obtemos,

$$c_1 = \frac{4}{ab}\int_0^a \, sen\,(i\pi\xi/a)d\xi \int_0^b \, sen\,(j\pi\eta/b)h(\xi,\eta)d\eta\,,$$
$$c_2 = \frac{1}{k_{ij}v}\frac{4}{ab}\int_0^a \, sen\,(i\pi\xi/a)d\xi \int_0^b \, sen\,(j\pi\eta/b)w(\xi,\eta)d\eta\,.$$

As expansões de h e w são então,

$$\begin{aligned} h(x,y) &= \sum_{ij} sen\,(i\pi x/a)\, sen\,(j\pi y/b) \times \\ &\quad \times \frac{4}{ab}\int_0^a \, sen\,(i\pi\xi/a)d\xi \int_0^b \, sen\,(j\pi\eta/b)h(\xi,\eta)d\eta\,, \\ w(x,y) &= \sum_{ij} sen\,(i\pi x/a)\, sen\,(j\pi y/b) \times \\ &\quad \times \frac{4}{ab}\int_0^a \, sen\,(i\pi\xi/a)d\xi \int_0^b \, sen\,(j\pi\eta/b)w(\xi,\eta)d\eta\,. \end{aligned}$$

Se h e w são constantes h_0 e w_0,

$$
\begin{aligned}
h_0 &= \sum_{ij} sen\,(i\pi x/a)\, sen\,(j\pi y/b) \times \\
&\quad \times \frac{4}{ab} h_0 \int_0^a sen\,(i\pi\xi/a) d\xi \int_0^b sen\,(j\pi\eta/b) d\eta\,, \\
&= \sum_{ij} sen\,[(2i-1)\pi x/a]\, sen\,[(2j-1)\pi y/b] \times \\
&\quad \times \frac{4}{ab} h_0 \frac{2a}{(2i-1)\pi} \frac{2b}{(2j-1)\pi}\,, \\
w_0 &= \sum_{ij} sen\,(i\pi x/a)\, sen\,(j\pi y/b) \times \\
&\quad \times \frac{4}{ab} w_0 \int_0^a sen\,(i\pi\xi/a) d\xi \int_0^b sen\,(j\pi\eta/b) d\eta\,, \\
&= \sum_{ij} sen\,[(2i-1)\pi x/a]\, sen\,[(2j-1)\pi y/b] \times \\
&\quad \times \frac{4}{ab} w_0 \frac{2a}{(2i-1)\pi} \frac{2b}{(2j-1)\pi}\,,
\end{aligned}
$$

em que usamos,

$$
\int_0^a sen\,(i\pi\xi/a) d\xi = \begin{cases} \dfrac{2a}{i\pi}\,, & i \text{ ímpar}\,, \\ 0\,, & i \text{ par}\,. \end{cases}
$$

Das relações acima obtemos,

$$
1 = \frac{16}{\pi^2} \sum_{ij} \frac{sen\,[(2i-1)\pi x/a]}{2i-1} \frac{sen\,[(2j-1)\pi y/b]}{2j-1}\,. \qquad (13)
$$

A figura 1 mostra um gráfico da função acima para alguns termos da série.

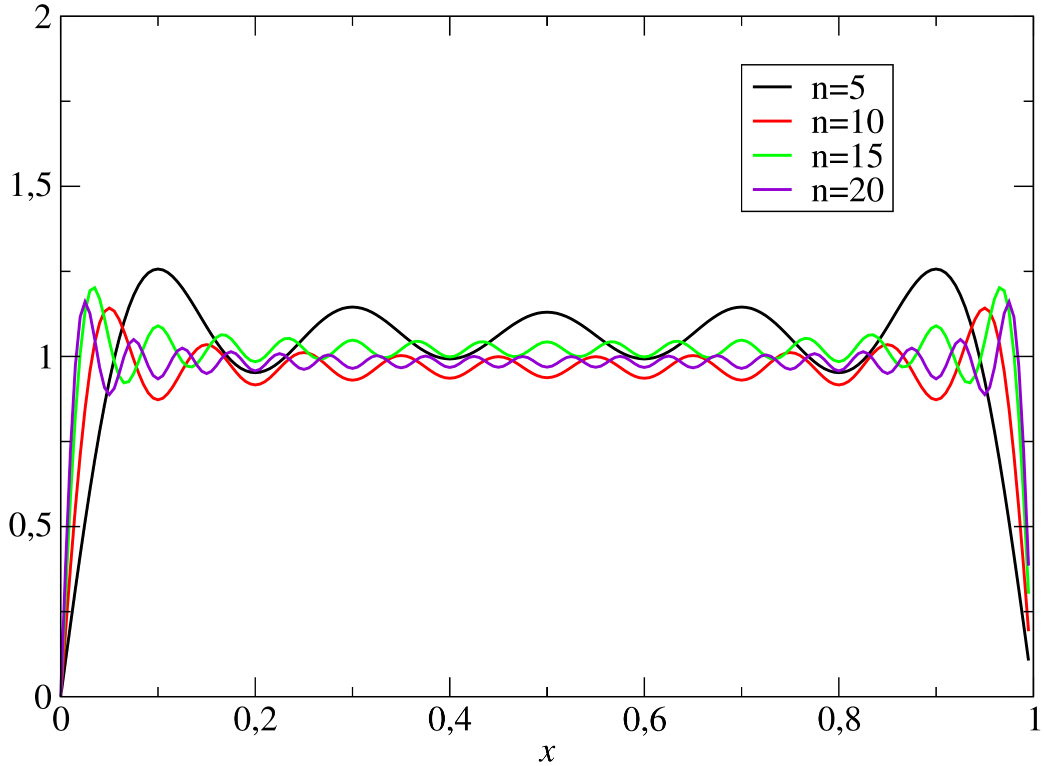

Fig. 1. Gráfico da equação (13) para os n primeiros termos da série, com $n = 5, 10, 15, 20$. *Escolhemos* $a = b = 1$, $y = 0, 5$.

A solução é assim,

$$\begin{aligned}
u(x,y,t) &= \sum_{ij} sen\,(i\pi x/a)\, sen\,(j\pi y/b) \cos(k_{ij}vt) \times \\
&\times \frac{4}{ab} \int_0^a sen\,(i\pi\xi/a) d\xi \int_0^b sen\,(j\pi\eta/b) h(\xi,\eta) d\eta \\
&+ \sum_{ij} sen\,(i\pi x/a)\, sen\,(j\pi y/b)\, sen\,(k_{ij}vt) \times \\
&\times \frac{1}{k_{ij}v} \frac{4}{ab} \int_0^a sen\,(i\pi\xi/a) d\xi \int_0^b sen\,(j\pi\eta/b) w(\xi,\eta) d\eta \,, \\
&\alpha_i = i\pi/a\,, \quad \mu_j = j\pi/b\,, \quad i,j = 1,2,\ldots \\
&k_{ij}^2 = \mu_j^2 + \alpha_i^2\,, \quad k^2 > \alpha^2\,.
\end{aligned}$$

3. Calcule $u(x,y,t)$ em $0 < x < a$, $0 < y < b$, $t > 0$ com,

$$
\begin{aligned}
u(0,y,t) = f_1(y)\,, \quad u(a,y,t) = f_2(y)\,, \\
u(x,0,t) = g_1(x)\,, \quad u(x,b,t) = g_2(x)\,.
\end{aligned}
$$

A posição inicial é dada por $h(x, y)$ e a velocidade inicial por $w(x, y)$.

Escrevemos a solução como,

$$
u(x,y,t) = u_1(x,y,t) + u_2(x,y,t) + u_3(x,y,t) + u_4(x,y,t)\,,
$$

com as condições de contorno,

$$
\begin{aligned}
u_1(0,y,t) &= f_1(y)\,, \\
u_2(a,y,t) &= f_2(y)\,, \\
u_3(x,0,t) &= g_1(x)\,, \\
u_4(x,b,t) &= g_2(x)\,,
\end{aligned}
$$

e $u_1(a, y, 0) = 0$ *etc. Consideramos as condições iniciais satisfeitas apenas por* u_1.

(a) Cálculo de u_1.

Escrevemos u_1 *como,*

$$
\begin{aligned}
u_1(x,y,t) &= u_e(x,y) + \sum_{k,\alpha} [a_1 \cos(\alpha x) + a_2\, sen\,(\alpha x)] \times \\
&\quad \times [b_1 \cos(\mu y) + b_2\, sen\,(\mu y)] \times \\
&\quad \times [c_1 \cos(kvt) + c_2\, sen\,(kvt)]\,, \\
&\quad \mu = \sqrt{k^2 - \alpha^2}\,, \quad k^2 > \alpha^2\,,
\end{aligned}
$$

em que u_e *é a solução da equação estacionária, a equação de Laplace. As condições de contorno nos dão,*

$$
\begin{aligned}
u_1(0,y,t) &= f_1(y) = u_e(0,y) + \sum_{k,\alpha} a_1[b_1\cos(\mu y) + b_2\, sen\,(\mu y)] \times \\
&\quad \times[c_1\cos(kvt) + c_2\, sen\,(kvt)]\,, \\
u_1(a,y,t) &= 0 = u_e(a,y) + \sum_{k,\alpha}[a_1\cos(\alpha a) + a_2\, sen\,(\alpha a)] \times \\
&\quad \times[b_1\cos(\mu y) + b_2\, sen\,(\mu y)] \times \\
&\quad \times[c_1\cos(kvt) + c_2\, sen\,(kvt)]\,, \\
u_1(x,0,t) &= 0 = u_e(x,0) + \sum_{k,\alpha}[a_1\cos(\alpha x) + a_2\, sen\,(\alpha x)]b_1 \times \\
&\quad \times[c_1\cos(kvt) + c_2\, sen\,(kvt)]\,, \\
u_1(x,b,t) &= 0 = u_e(x,b) + \sum_{k,\alpha}[a_1\cos(\alpha x) + a_2\, sen\,(\alpha x)] \times \\
&\quad \times[b_1\cos(\mu b) + b_2\, sen\,(\mu b)] \times \\
&\quad \times[c_1\cos(kvt) + c_2\, sen\,(kvt)]\,, \\
&\quad \mu = \sqrt{k^2 - \alpha^2}\,, \;\; k^2 > \alpha^2\,.
\end{aligned}
$$

Satisfazemos as condições acima escolhendo,

$$
\begin{aligned}
&a_1 = b_1 = 0\,, \\
&\alpha_i a = i\pi\,, \;\; i = 1, 2, \ldots \\
&\mu_j b = j\pi\,, \;\; j = 1, 2, \ldots
\end{aligned}
$$

A condições acima ficam então,

$$
\begin{aligned}
&f_1(y) = u_e(0,y)\,, \\
&0 = u_e(a,y)\,, \\
&0 = u_e(x,0)\,, \\
&0 = u_e(x,b)\,.
\end{aligned}
$$

A solução u_1 fica então,

$$
\begin{aligned}
u_1(x,y,t) &= u_e(x,y) + \sum_{ij} a_2 b_2\, sen\,(i\pi x/a)\, sen\,(j\pi y/b) \times \\
&\quad \times[c_1\cos(k_{ij}vt) + c_2\, sen\,(k_{ij}vt)]\,, \\
&\quad \mu_j^2 + \alpha_i^2 = k_{ij}^2\,, \;\; k^2 > \alpha^2\,.
\end{aligned}
$$

Usando agora as condições iniciais temos,

$$
\begin{aligned}
u_1(x,y,0) &= h(x,y)\,, \\
&= u_e(x,y) + \sum_{ij} a_2 b_2 c_1 \, sen\,(i\pi x/a)\, sen\,(j\pi y/b)\,, \\
\frac{\partial u_1(x,y,0)}{\partial t} &= w(x,y)\,, \\
&= \sum_{ij} a_2 b_2 c_2 k_{ij} v \, sen\,(i\pi x/a)\, sen\,(j\pi y/b)\,.
\end{aligned}
$$

Temos séries duplas de Fourier de senos, logo,

$$
\begin{aligned}
\sum_i a_2 b_2 c_1 \, sen\,(i\pi x/a) = \frac{2}{b}\int_0^b [h(x,\eta) - u_e(x,\eta)]\, sen\,(j\pi\eta/b) d\eta\,, \\
\sum_i a_2 b_2 c_2 k_{ij} v \, sen\,(i\pi x/a) = \frac{2}{b}\int_0^b w(x,\eta)\, sen\,(j\pi\eta/b) d\eta\,.
\end{aligned}
$$

Prosseguindo,

$$
\begin{aligned}
a_2 b_2 c_1 &= \frac{2}{a}\int_0^a d\xi \, sen\,(i\pi\xi/a) \times \\
&\quad \times \frac{2}{b}\int_0^b [h(\xi,\eta) - u_e(\xi,\eta)]\, sen\,(j\pi\eta/b) d\eta\,, \\
a_2 b_2 c_2 k_{ij} v &= \frac{2}{a}\int_0^a d\xi \, sen\,(i\pi\xi/a) \times \\
&\quad \times \frac{2}{b}\int_0^b w(\xi,\eta)\, sen\,(j\pi\eta/b) d\eta\,.
\end{aligned}
$$

Portanto,

$$
\begin{aligned}
h(x,y)-u_e(x,y) &= \sum_{ij} sen\,(i\pi x/a)\, sen\,(j\pi y/b) \times \\
&\times \frac{2}{a}\int_0^a d\xi\, sen\,(i\pi\xi/a) \times \\
&\times \frac{2}{b}\int_0^b [h(\xi,\eta)-u_e(\xi,\eta)]\, sen\,(j\pi\eta/b) d\eta\,, \\
w(x,y) &= \sum_{ij} sen\,(i\pi x/a)\, sen\,(j\pi y/b) \times \\
&\times \frac{2}{a}\int_0^a d\xi\, sen\,(i\pi\xi/a) \times \\
&\times \frac{2}{b}\int_0^b w(\xi,\eta)\, sen\,(j\pi\eta/b) d\eta\,.
\end{aligned}
$$

Se $h-u_e$ *e* w *são constantes* h_0 *e* w_0*,*

$$
\begin{aligned}
h_0 &= \sum_{ij} sen\,(i\pi x/a)\, sen\,(j\pi y/b) \times \\
&\times \frac{2}{a}\int_0^a d\xi\, sen\,(i\pi\xi/a) \frac{2}{b}\int_0^b h_0\, sen\,(j\pi\eta/b) d\eta\,, \\
w_0 &= \sum_{ij} sen\,(i\pi x/a)\, sen\,(j\pi y/b) \times \\
&\times \frac{2}{a}\int_0^a d\xi\, sen\,(i\pi\xi/a) \frac{2}{b}\int_0^b w_0\, sen\,(j\pi\eta/b) d\eta\,,
\end{aligned}
$$

ou,

$$
\begin{aligned}
1 &= \frac{16}{\pi^2}\sum_{ij} \frac{sen\,[(2i-1)\pi x/a]}{2i-1}\, \frac{sen\,[(2j-1)\pi y/b]}{2j-1} \times \\
1 &= \frac{16}{\pi^2}\sum_{ij} \frac{sen\,[(2i-1)\pi x/a]}{2i-1}\, \frac{sen\,[(2j-1)\pi y/b]}{2j-1}\,,
\end{aligned}
$$

em que usamos,

$$
\int_0^a sen\,(i\pi\xi/a) d\xi = \begin{cases} \dfrac{2a}{i\pi}\,, & i\ \acute{\imath}mpar\,, \\ 0\,, & i\ par\,. \end{cases}
$$

As expressões acima concordam com (13).

A solução fica então,

$$
\begin{aligned}
u_1(x,y,t) &= u_e(x,y) \\
&+\sum_{ij} sen\,(i\pi x/a)\, sen\,(j\pi y/b) \cos(k_{ij}vt) \times \\
&\times \frac{2}{a}\int_0^a d\xi\, sen\,(i\pi\xi/a) \times \\
&\times \frac{2}{b}\int_0^b [h(\xi,\eta) - u_e(\xi,\eta)]\, sen\,(j\pi\eta/b) d\eta\,, \\
&+\sum_{ij} sen\,(i\pi x/a)\, sen\,(j\pi y/b)\, sen\,(k_{ij}vt) \times \\
&\times \frac{1}{k_{ij}v}\frac{2}{a}\int_0^a d\xi\, sen\,(i\pi\xi/a) \times \\
&\times \frac{2}{b}\int_0^b w(\xi,\eta)\, sen\,(j\pi\eta/b) d\eta\,, \\
&\mu_j^2 + \alpha_i^2 = k_{ij}^2\,, \quad k^2 > \alpha^2\,.
\end{aligned}
$$

Se $h = w = 0$*,*

$$
\begin{aligned}
u_1(x,y,t) &= u_e(x,y) \\
&-\sum_{ij} sen\,(i\pi x/a)\, sen\,(j\pi y/b) \cos(k_{ij}vt) \times \\
&\times \frac{2}{a}\int_0^a d\xi\, sen\,(i\pi\xi/a) \frac{2}{b}\int_0^b u_e(\xi,\eta)\, sen\,(j\pi\eta/b) d\eta\,, \\
&\mu_j^2 + \alpha_i^2 = k_{ij}^2\,, \quad k^2 > \alpha^2\,.
\end{aligned}
$$

Em $t = 0$ *o lado direito da expressão acima se anula, como deve ser se temos* $h = 0$*. Para o cálculo de* u_2*, etc. procedemos de forma análoga.*

4. Calcule $u(x,y,t)$ em $0 < x < a$, $0 < y < b$, $t > 0$ com,

$$
\begin{aligned}
u(0,y,t) = f_1(y,t)\,, \quad u(a,y,t) = f_2(y,t)\,, \\
u(x,0,t) = g_1(x,t)\,, \quad u(x,b,t) = g_2(x,t)\,.
\end{aligned}
$$

A posição inicial é dada por $h(x,y)$ e a velocidade inicial por $w(x,y)$.

Escrevemos a solução como,

$$u(x,y,t) = u_1(x,y,t) + u_2(x,y,t) + u_3(x,y,t) + u_4(x,y,t)\,,$$

com as condições de contorno,

$$\begin{aligned} u_1(0,y,t) &= f_1(y,t)\,,\\ u_2(a,y,t) &= f_2(y,t)\,,\\ u_3(x,0,t) &= g_1(x,t)\,,\\ u_4(x,b,t) &= g_2(x,t)\,, \end{aligned}$$

e $u_1(a,y,t) = 0$ *etc. Supomos também,*

$$u_1(x,y,0) = h(x,y)\,,\quad \frac{\partial u_1(x,y,0)}{\partial t} = w(x,y)\,,$$

com $u_2(x,y,0) = 0$*, etc.*

(a) Cálculo de u_1*.*

Escrevemos u_1 *como,*

$$\begin{aligned} u_1(x,y,t) &= \sum_{k,\alpha}[a_1\cos(\alpha x) + a_2\,sen\,(\alpha x)][b_1\cos(\mu y) + b_2\,sen\,(\mu y)]\times\\ &\quad \times[c_1\cos(kvt) + c_2\,sen\,(kvt)]\,,\quad \mu = \sqrt{k^2-\alpha^2}\,,\quad k^2 > \alpha^2\,. \end{aligned}$$

Agora não escrevemos o termo u_e *porque as condições de contorno dependem do tempo. Temos,*

$$\begin{aligned} u_1(0,y,t) &= f_1(y,t) = \sum_{k,\alpha} a_1[b_1\cos(\mu y) + b_2\,sen\,(\mu y)]\times\\ &\quad \times[c_1\cos(kvt) + c_2\,sen\,(kvt)]\,,\\ u_1(a,y,t) &= 0 = \sum_{k,\alpha}[a_1\cos(\alpha a) + a_2\,sen\,(\alpha a)][b_1\cos(\mu y) + b_2\,sen\,(\mu y)]\times\\ &\quad \times[c_1\cos(kvt) + c_2\,sen\,(kvt)]\,,\\ u_1(x,0,t) &= 0 = \sum_{k,\alpha}[a_1\cos(\alpha x) + a_2\,sen\,(\alpha x)]b_1\times\\ &\quad \times[c_1\cos(kvt) + c_2\,sen\,(kvt)]\,,\\ u_1(x,b,t) &= 0 = \sum_{k,\alpha}[a_1\cos(\alpha x) + a_2\,sen\,(\alpha x)][b_1\cos(\mu b) + b_2\,sen\,(\mu b)]\times\\ &\quad \times[c_1\cos(kvt) + c_2\,sen\,(kvt)]\,,\\ &\quad \mu = \sqrt{k^2-\alpha^2}\,,\quad k^2 > \alpha^2\,. \end{aligned}$$

Satisfazemos as condições acima escolhendo,

$$
\begin{aligned}
&a_2 = b_1 = 0\,,\\
&\mu_i b = i\pi\,,\quad i = 1, 2, \ldots\\
&\alpha_j a = (2j-1)\pi/2 \quad j = 1, 2, \ldots
\end{aligned}
$$

A condição para f_1 fica então, fazendo $a_1 = b_2 = 1$,

$$
\begin{aligned}
f_1(y,t) &= \sum_{ij} sen\,(i\pi y/b)[c_1 \cos(k_{ij} v t) + c_2\, sen\,(k_{ij} v t)]\,,\\
& k_{ij}^2 = \mu_i^2 + \alpha_j^2\,,\quad k^2 > \alpha^2\,,
\end{aligned}
$$

e a solução é,

$$
\begin{aligned}
u_1(x,y,t) &= \sum_{ij} \cos(\alpha_j x)\, sen\,(\mu_i y) \times\\
&\quad \times [c_1 \cos(k_{ij} v t) + c_2\, sen\,(k_{ij} v t)]\,,\\
&\quad \mu_i = \sqrt{k_{ij}^2 - \alpha_j^2}\,,\quad k^2 > \alpha^2\,.
\end{aligned}
$$

Em $t = 0$,

$$
\begin{aligned}
u_1(x,y,0) &= h(x,y) = \sum_{ij} \cos(\alpha_j x)\, sen\,(i\pi y/b) c_1\,,\\
\frac{\partial u_1(x,y,0)}{\partial t} &= w(x,y) = \sum_{ij} \cos(\alpha_j x)\, sen\,(i\pi y/b) c_2 k_{ij} v\,,\\
& \mu_i^2 + \alpha_j^2 = k_{ij}^2\,,\quad k^2 > \alpha^2\,,
\end{aligned}
$$

portanto,

$$
\begin{aligned}
&\sum_j c_1 \cos(\alpha_j x) = \frac{2}{b} \int_0^b sen\,(i\pi\eta/b) h(x,\eta) d\eta\,,\\
&\sum_j c_2 k_{ij} v \cos(\alpha_j x) = \frac{2}{b} \int_0^b sen\,(i\pi\eta/b) w(x,\eta) d\eta\,.
\end{aligned}
$$

Os coeficientes das expansões acima são,

$$c_{1l} = \frac{2}{a}\int_0^a d\xi \cos(\alpha_l \xi)\frac{2}{b}\int_0^b \, sen\,(i\pi\eta/b)h(\xi,\eta)d\eta\,,$$
$$c_{2l} = \frac{1}{k_{il}v}\frac{2}{a}\int_0^a d\xi \cos(\alpha_l \xi)\frac{2}{b}\int_0^b \, sen\,(i\pi\eta/b)w(\xi,\eta)d\eta\,.$$

A expansão de f_1 fica então,

$$\begin{aligned} f_1(y,t) &= \sum_{ij} sen\,(i\pi y/b)\cos(k_{ij}vt) \times \\ &\times \frac{2}{a}\int_0^a d\xi \cos(\alpha_j \xi)\frac{2}{b}\int_0^b \, sen\,(i\pi\eta/b)h(\xi,\eta)d\eta \\ &+ \sum_{ij} sen\,(i\pi y/b)\; sen\;(k_{ij}vt) \times \\ &\times \frac{1}{k_{ij}v}\frac{2}{a}\int_0^a d\xi \cos(\alpha_j \xi)\frac{2}{b}\int_0^b \, sen\,(i\pi\eta/b)w(\xi,\eta)d\eta\,, \\ & k_{ij}^2 = \mu_i^2 + \alpha_j^2\,, \quad k^2 > \alpha^2\,. \end{aligned}$$

Notemos que f_1, h e w não são completamente arbitrárias, como no problema anterior.

A solução é,

$$\begin{aligned} u_1(x,y,t) &= \sum_{ij} \cos(\alpha_j x)\; sen\,(\mu_i y)\cos(k_{ij}vt) \times \\ &\times \frac{2}{a}\int_0^a d\xi \cos(\alpha_j \xi)\frac{2}{b}\int_0^b \, sen\,(i\pi\eta/b)h(\xi,\eta)d\eta\,, \\ &+ \sum_{ij} \cos(\alpha_j x)\; sen\,(\mu_i y)\; sen\;(k_{ij}vt) \times \\ &\times \frac{1}{k_{ij}v}\frac{2}{a}\int_0^a d\xi \cos(\alpha_j \xi)\frac{2}{b}\int_0^b \, sen\,(i\pi\eta/b)w(\xi,\eta)d\eta\,, \\ & \mu_i^2 + \alpha_j^2 = k_{ij}^2\,, \quad k^2 > \alpha^2\,. \end{aligned}$$

5. Determine $u(t,x,y)$ para os seguintes problemas:
(a)
$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2} + f(x,y)\,,$$

(b)

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2} + \alpha^2 \,,$$

com α constante,

(c)

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2} + \alpha^2 u \,,$$

com α constante e,

$$\begin{aligned}
&0 \leq x \leq a \,, \;\; 0 \leq x \leq b \,, \\
&u(t,0,y) = \mu_1(y) \,, \;\; u(t,a,y) = \mu_2(y) \,, \\
&u(t,x,0) = \nu_1(x) \,, \;\; u(t,x,b) = \nu_2(x) \,, \\
&u(0,x,y) = \varphi(x,y) \,, \;\; u_t(0,x,y) = \psi(x,y) \,.
\end{aligned}$$

6. Determine $u(t,x,y)$ para os seguintes problemas:

(a)

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2} + f(t,x,y) \,,$$

(b)

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2} + \alpha^2 \,,$$

com α constante,

(c)

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2} + \alpha^2 u \,,$$

com α constante e,

$$\begin{aligned}
&0 \leq x \leq a \,, \;\; 0 \leq x \leq b \,, \\
&u(t,0,y) = \mu_1(t,y) \,, \;\; u(t,a,y) = \mu_2(t,y) \,, \\
&u(t,x,0) = \nu_1(t,x) \,, \;\; u(t,x,b) = \nu_2(t,x) \,, \\
&u(0,x,y) = \varphi(x,y) \,, \;\; u_t(0,x,y) = \psi(x,y) \,.
\end{aligned}$$

Considerando $F(x,y,z)$

A equação (5) fica,

$$\frac{d^2F}{dx^2} + \frac{d^2F}{dy^2} + \frac{d^2F}{dz^2} + k^2 F = 0 \,. \tag{14}$$

Substituindo $F(x,y,z) = X(x)Y(y)Z(z)$,

$$
\begin{aligned}
&X''YZ + XY''Z + XYZ'' + k^2XYZ = 0\,,\\
&\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} + k^2 = 0\,,
\end{aligned}
$$

ou,

$$\frac{X''}{X} + \frac{Y''}{Y} = -\frac{Z''}{Z} - k^2 = -\alpha^2\,.$$

Obtemos assim a seguinte equação para Z,

$$Z'' + (k^2 - \alpha^2)Z = 0\,.$$

Para X e Y obtemos,

$$
\begin{aligned}
&\frac{X''}{X} + \frac{Y''}{Y} = -\alpha^2\,,\\
&\frac{X''}{X} = -\frac{Y''}{Y} - \alpha^2 = -\mu^2\,,
\end{aligned}
$$

e as equações para X e Y são,

$$
\begin{aligned}
&X'' + \mu^2 X = 0\,,\\
&Y'' + (\alpha^2 - \mu^2)Y = 0\,.
\end{aligned}
$$

As soluções X, Y, Z são portanto,

$$
\begin{aligned}
&X(x) = a_1\cos(\mu x) + a_2\,\mathrm{sen}\,(\mu x)\,,\\
&Y(y) = \begin{cases} b_1\cos(\beta y) + b_2\,\mathrm{sen}\,(\beta y)\,, & \beta^2 = \alpha^2 - \mu^2 > 0\,,\\ c_1\cosh(\beta y) + c_2\,\mathrm{senh}\,(\beta y)\,, & \beta^2 = \mu^2 - \alpha^2 > 0\,,\end{cases}\\
&Z(z) = \begin{cases} d_1\cos(\gamma z) + d_2\,\mathrm{sen}\,(\gamma z)\,, & \gamma^2 = k^2 - \alpha^2 > 0\,,\\ e_1\cosh(\gamma z) + e_2\,\mathrm{senh}\,(\gamma z)\,, & \gamma^2 = \alpha^2 - k^2 > 0\,.\end{cases}
\end{aligned}
\tag{15}
$$

Também podemos ter,

$$\frac{X''}{X} + \frac{Y''}{Y} = -\frac{Z''}{Z} - k^2 = +\alpha^2\,.$$

A equação para Z agora é,

$$Z'' + (k^2 + \alpha^2)Z = 0\,.$$

Para X e Y obtemos,

$$\frac{X''}{X} + \frac{Y''}{Y} = +\alpha^2\,,$$
$$\frac{X''}{X} - \alpha^2 = -\frac{Y''}{Y} = +\mu^2\,.$$

Portanto,

$$X'' - (\alpha^2 + \mu^2)X = 0\,,$$
$$Y'' + \mu^2 Y = 0\,.$$

As soluções são então,

$$\begin{aligned} X(x) &= a_1 \cosh(\beta x) + a_2 \,\text{senh}\,(\beta x)\,, \quad \beta^2 = \alpha^2 + \mu^2\,, \\ Y(y) &= b_1 \cos(\mu y) + b_2 \,\text{sen}\,(\mu y)\,, \\ Z(z) &= c_1 \cos(\gamma z) + c_2 \,\text{sen}\,(\gamma z)\,, \quad \gamma^2 = k^2 + \alpha^2\,. \end{aligned} \tag{16}$$

Outra possibilidade é escrevermos,

$$\frac{Y''}{Y} = -\frac{X''}{X} - \frac{Z''}{Z} - k^2 = +\alpha^2\,.$$

A equação para Y é assim,

$$Y'' - \alpha^2 Y = 0\,.$$

Para X temos,

$$-\frac{X''}{X} - k^2 = +\alpha^2 + \frac{Z''}{Z} = +\mu^2\,,$$
$$-\frac{X''}{X} = \mu^2 + k^2\,,$$
$$X'' + (\mu^2 + k^2)X = 0\,,$$

e para Z,

$$
\begin{aligned}
&\frac{Z''}{Z} = \mu^2 - \alpha^2\,,\\
&Z'' + (\alpha^2 - \mu^2)Z = 0\,.
\end{aligned}
$$

Portanto,

$$
\begin{aligned}
&X(x) = a_1\cos(\beta x) + a_2\,\text{sen}\,(\beta x)\,, \quad \beta^2 = \mu^2 + k^2\,,\\
&Y(y) = b_1\cosh(\alpha y) + b_2\,\text{senh}\,(\alpha y)\,,\\
&Z(z) = \begin{cases} c_1\cos(\gamma z) + c_2\,\text{sen}\,(\gamma z)\,, & \gamma^2 = \alpha^2 - \mu^2 > 0\,,\\ d_1\cosh(\delta z) + d_2\,\text{senh}\,(\delta z)\,, & \gamma^2 = \mu^2 - \alpha^2 > 0\,. \end{cases}
\end{aligned}
\tag{17}
$$

Escrevendo agora,

$$
\begin{aligned}
&\frac{Y''}{Y} = -\frac{X''}{X} - \frac{Z''}{Z} - k^2 = -\alpha^2\,,\\
&-\frac{X''}{X} = -\alpha^2 + \frac{Z''}{Z} + k^2 = -\mu^2\,,\\
&\frac{Z''}{Z} = -\mu^2 - k^2 + \alpha^2\,,
\end{aligned}
$$

obtemos as equações,

$$
\begin{aligned}
&X'' - \mu^2 X = 0\,,\\
&Y'' + \alpha^2 Y = 0\,,\\
&Z'' + (\mu^2 + k^2 - \alpha^2)Z = 0\,.
\end{aligned}
$$

As soluções são portanto,

$$
\begin{aligned}
&X(x) = a_1\cosh(\mu x) + a_2\,\text{senh}\,(\mu x)\,,\\
&Y(y) = b_1\cos(\alpha y) + b_2\,\text{sen}\,(\alpha y)\,,\\
&Z(z) = \begin{cases} c_1\cos(\beta z) + c_2\,\text{sen}\,(\beta z)\,, & \beta^2 = \mu^2 + k^2 - \alpha^2 > 0\\ d_1\cosh(\beta z) + d_2\,\text{senh}\,(\beta z)\,, & \beta^2 = \alpha^2 - \mu^2 - k^2 > 0\,, \end{cases}
\end{aligned}
\tag{18}
$$

Por fim, escrevemos,

$$
\begin{aligned}
&\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} + k^2 = 0\,, \\
&\frac{X''}{X} = -\frac{Y''}{Y} - \frac{Z''}{Z} - k^2 = +\alpha^2\,, \\
&\frac{Y''}{Y} = -\frac{Z''}{Z} - k^2 - \alpha^2 = +\mu^2\,, \\
&\frac{Z''}{Z} = -k^2 - \alpha^2 - \mu^2\,,
\end{aligned}
$$

ou,

$$
\begin{aligned}
&X'' - \alpha^2 X = 0\,, \\
&Y'' - \mu^2 Y = 0\,, \\
&Z'' + (k^2 + \alpha^2 + \mu^2)Z = 0\,.
\end{aligned}
$$

A solução é então,

$$
\begin{aligned}
&X(x) = a_1 \cosh(\alpha x) + a_2 \,\mathrm{senh}\,(\alpha x)\,, \\
&Y(y) = b_1 \cosh(\mu y) + b_2 \,\mathrm{senh}\,(\mu y)\,, \\
&Z(z) = c_1 \cos(\beta z) + c_2 \,\mathrm{sen}\,(\beta z)\,, \quad \beta^2 = k^2 + \alpha^2 + \mu^2 > 0\,.
\end{aligned}
$$

3 Problemas

1. Calcule $u(x, y, z, t)$ em $0 < x < a$, $0 < y < b$, $0 < z < c$, $t > 0$ com,

$$
\begin{aligned}
&u(0, y, z, t) = 0\,, \quad u(a, y, z, t) = 0\,, \\
&u(x, 0, z, t) = 0\,, \quad u(x, b, z, t) = 0\,, \\
&u(x, y, 0, t) = 0\,, \quad u(x, 0, c, t) = 0\,.
\end{aligned}
$$

A posição inicial é $w_1(x, y, z)$ e a velocidade inicial é $w_2(x, y, z)$.

2. Calcule $u(x, y, z, t)$ em $0 < x < a$, $0 < y < b$, $0 < z < c$, $t > 0$ com,

$$
\begin{aligned}
&u(0, y, z, t) = f_1(y, z)\,, \quad u(a, y, z, t) = f_2(y, z)\,, \\
&u(x, 0, z, t) = g_1(x, z)\,, \quad u(x, b, z, t) = g_2(x, z)\,, \\
&u(x, y, 0, t) = h_1(x, y)\,, \quad u(x, y, c, t) = h_2(x, y)\,.
\end{aligned}
$$

A posição inicial é $w_1(x,y,z)$ e a velocidade inicial é $w_2(x,y,z)$.

Escrevemos a solução como uma soma,

$$u = u_1 + u_2 + u_3 + u_4 + u_5 + u_6\,,$$

com as condições de contorno em $t = 0$*,*

$$\begin{aligned}
u_1(0,y,z,t) &= f_1(y,z)\,,\\
u_2(a,y,z,t) &= f_2(y,z)\,,\\
u_3(x,0,z,t) &= g_1(x,z)\,,\\
u_4(x,b,z,t) &= g_2(x,z)\,,\\
u_5(x,y,0,t) &= h_1(x,y)\,,\\
u_6(x,y,c,t) &= h_2(x,y)\,.
\end{aligned}$$

Nas outras faces as funções u_j *se anulam. Consideramos as condições iniciais satisfeitas por* u_1*.*

(a) Cálculo de u_1*.*

Escrevemos a solução como,

$$\begin{aligned}
u_1(x,y,z,t) \;\; &= \;\; u_e(x,y,z) + \sum_{k,\alpha,\mu} [a_1 \cos(\mu x) + a_2\, sen\,(\mu x)] \times\\
&\times [b_1 \cos(\beta y) + b_2\, sen\,(\beta y)] \times\\
&\times [c_1 \cos(\gamma z) + c_2\, sen\,(\gamma z)][d_1 \cos(kvt) + d_2\, sen\;(kvt)]\,,\\
&\beta^2 = \alpha^2 - \mu^2 > 0\,,\;\; \gamma^2 = k^2 - \alpha^2 > 0\,.
\end{aligned}$$

As condições de contorno nos dão,

$$
\begin{aligned}
u_1(0,y,z,t) &= f_1(y,z) = u_e(0,y,z) + \sum_{k,\alpha,\mu} a_1[b_1\cos(\beta y) + b_2\, sen\,(\beta y)] \times \\
&\quad \times[c_1\cos(\gamma z) + c_2\, sen\,(\gamma z)] \times \\
&\quad \times[d_1\cos(kvt) + d_2\, sen\ (kvt)]\,, \\
u_1(a,y,z,t) &= 0 = u_e(a,y,z) + \sum_{k,\alpha,\mu} [a_1\cos(\mu a) + a_2\, sen\,(\mu a)] \times \\
&\quad \times[b_1\cos(\beta y) + b_2\, sen\,(\beta y)] \times \\
&\quad \times[c_1\cos(\gamma z) + c_2\, sen\,(\gamma z)] \times \\
&\quad \times[d_1\cos(kvt) + d_2\, sen\ (kvt)]\,, \\
u_1(x,0,z,t) &= 0 = u_e(x,0,z) + \sum_{k,\alpha,\mu} [a_1\cos(\mu x) + a_2\, sen\,(\mu x)] b_1 \times \\
&\quad \times[c_1\cos(\gamma z) + c_2\, sen\,(\gamma z)] \times \\
&\quad \times[d_1\cos(kvt) + d_2\, sen\ (kvt)]\,, \\
u_1(x,b,z,t) &= 0 = u_e(x,b,z) + \sum_{k,\alpha,\mu} [a_1\cos(\mu x) + a_2\, sen\,(\mu x)] \times \\
&\quad \times[b_1\cos(\beta b) + b_2\, sen\,(\beta b)] \times \\
&\quad \times[c_1\cos(\gamma z) + c_2\, sen\,(\gamma z)] \times \\
&\quad \times[d_1\cos(kvt) + d_2\, sen\ (kvt)]\,, \\
u_1(x,y,0,t) &= 0 = u_e(x,y,0) + \sum_{k,\alpha,\mu} [a_1\cos(\mu x) + a_2\, sen\,(\mu x)] \times \\
&\quad \times[b_1\cos(\beta y) + b_2\, sen\,(\beta y)] c_1 \times \\
&\quad \times[d_1\cos(kvt) + d_2\, sen\ (kvt)]\,, \\
u_1(x,y,c,t) &= 0 = u_e(x,y,c) + \sum_{k,\alpha,\mu} [a_1\cos(\mu x) + a_2\, sen\,(\mu x)] \times \\
&\quad \times[b_1\cos(\beta y) + b_2\, sen\,(\beta y)] \times \\
&\quad \times[c_1\cos(\gamma c) + c_2\, sen\,(\gamma c)] \times \\
&\quad \times[d_1\cos(kvt) + d_2\, sen\ (kvt)]\,, \\
&\quad \beta^2 = \alpha^2 - \mu^2 > 0\,, \;\; \gamma^2 = k^2 - \alpha^2 > 0\,,
\end{aligned}
$$

*****

Satisfazemos as equações acima escolhendo,

$$
\begin{aligned}
&b_1 = c_1 = a_2 = d_2 = 0\,, \\
&d_1 = 1\,, \\
&\mu_i a = (2i-1)\pi/2\,, \quad i = 1, 2, \ldots \\
&\beta_j b = j\pi\,, \quad j = 1, 2, \ldots \\
&\gamma_l c = l\pi\,, \quad l = 1, 2, \ldots
\end{aligned}
$$

A condição para f_1 fica,

$$
\begin{aligned}
&f_1(y,z) = \sum_{jl} a_1 b_2 c_2 d_1\, sen\,(j\pi y/b)\, sen\,(l\pi z/c)\,, \\
&\beta_j^2 = \alpha^2 - \mu_i^2 > 0\,, \quad \gamma_l^2 = k^2 - \alpha^2 > 0\,,
\end{aligned}
$$

portanto,

$$
\sum_j a_1 b_2 c_2 d_1\, sen\,(j\pi y/b) = \frac{2}{c}\int_0^c f_1(y,\eta)\, sen\,(l\pi\eta/c) d\eta\,.
$$

A expressão acima é outra expansão em série de senos, logo,

$$
a_1 b_2 c_2 d_1 \;=\; \frac{4}{bc}\int_0^b sen\,(j\pi\xi/b) d\xi \int_0^c sen\,(l\pi\eta/c) f_1(y,\eta) d\eta\,.
$$

A expansão para f_1 é assim,

$$
\begin{aligned}
f_1(y,z) \;=\;& \sum_{jl} sen\,(j\pi y/b)\, sen\,(l\pi z/c)\, \times \\
& \times \frac{4}{bc}\int_0^b sen\,(j\pi\xi/b) d\xi \int_0^c sen\,(l\pi\eta/c) f_1(y,\eta) d\eta\,, \\
& \beta_j^2 = \alpha^2 - \mu_i^2 > 0\,, \quad \gamma_l^2 = k^2 - \alpha^2 > 0\,, \\
& \mu_i = (2i-1)\pi/2a\,, \quad \beta_j = j\pi/b\,, \\
& \gamma_l = l\pi/c\,, \quad i, j, l = 1, 2, \ldots
\end{aligned}
$$

A solução u_1 é então,

$$\begin{aligned} u_1(x,y,z,t) &= \sum_{ijlk} \cos[(2i-1)\pi x/2a]\, sen\,(j\pi y/b)\, sen\,(l\pi z/c) \cos(kvt) \times \\ &\times \frac{4}{bc} \int_0^b sen\,(j\pi\xi/b) d\xi \int_0^c sen\,(l\pi\eta/c) f_1(y,\eta) d\eta\,, \\ &\beta_j^2 = \alpha^2 - \mu_i^2 > 0\,,\ \ \gamma_l^2 = k^2 - \alpha^2 > 0\,, \\ &\mu_i = (2i-1)\pi/2a\,,\ \ \beta_j = j\pi/b\,, \\ &\gamma_l = l\pi/c\,,\ \ i,j,l = 1,2,\ldots \end{aligned}$$

Para calcular u_2 *etc. seguimos um procedimento análogo.*

3. Calcule $u(x,y,z,t)$ em $0 < x < a$, $0 < y < b$, $0 < z < c$, $t > 0$ com,

$$\begin{aligned} u(0,y,z,t) = f_1(y,z,t)\,,\ \ u(a,y,z,t) = f_2(y,z,t)\,, \\ u(x,0,z,t) = g_1(x,z,t)\,,\ \ u(x,b,z,t) = g_2(x,z,t)\,, \\ u(x,y,0,t) = h_1(x,y,t)\,,\ \ u(x,y,c,t) = h_2(x,y,t)\,. \end{aligned}$$

A posição inicial é $w_1(x,y,z)$ e a velocidade inicial é $w_2(x,y,z)$.

4. Calcule $u(x,y,z,t)$ em $0 < x < a$, $t > 0$ com,

$$u(x,y,z,0) = h(x,y,z)\,,\ \ \frac{\partial u(x,y,z,0)}{\partial t} = w(x,y,z)\,.$$

5. Calcule $u(x,y,z,t)$ em $-\infty < x,y,z < +\infty$, $t > 0$ com,

$$u(x,y,z,0) = f(x,y,z)\,,\ \ \frac{\partial u(x,y,z,0)}{\partial t} = g(x,y,z)\,.$$

Escrevemos a solução, usando o princípio da superposição,

$$\begin{aligned} u(x,y,z,t) &= \int_0^\infty dk \int_0^\infty d\alpha \int_0^\infty d\mu \times \\ &\times [a_1 \cos(\mu x) + a_2\, sen\,(\mu x)] \times \\ &\times [b_1 \cos(\beta y) + b_2\, sen\,(\beta y)] \times \\ &\times [c_1 \cos(\gamma z) + c_2\, sen\,(\gamma z)][d_1 \cos(kvt) + d_2\, sen\ (kvt)]\,, \\ &\beta^2 = \alpha^2 - \mu^2 > 0\,,\ \ \gamma^2 = k^2 - \alpha^2 > 0\,. \end{aligned}$$

Podemos integrar em vez de somar porque as variáveis x,y,z *variam de* $-\infty$ *a* $+\infty$*. As condições de contorno ficam,*

$$
\begin{aligned}
u(x,y,z,0) &= f(x,y,z) = \int_0^\infty dk \int_0^\infty d\alpha \int_0^\infty d\mu \times \\
&\quad \times [a_1 \cos(\mu x) + a_2\, sen\,(\mu x)] \times \\
&\quad \times [b_1 \cos(\beta y) + b_2\, sen\,(\beta y)] \times \\
&\quad \times [c_1 \cos(\gamma z) + c_2\, sen\,(\gamma z)] d_1\,, \\
\frac{\partial u(x,y,z,0)}{\partial t} &= g(x,y,z) = \int_0^\infty dk \int_0^\infty d\alpha \int_0^\infty d\mu \times \\
&\quad \times [a_1 \cos(\mu x) + a_2\, sen\,(\mu x)] \times \\
&\quad \times [b_1 \cos(\beta y) + b_2\, sen\,(\beta y)] \times \\
&\quad \times [c_1 \cos(\gamma z) + c_2\, sen\,(\gamma z)] d_2 k v\,.
\end{aligned}
$$

Como não temos nenhuma outra informação, escolhemos $a_1 = b_1 = c_1 = 0$*,*

$$
\begin{aligned}
u(x,y,z,0) &= f(x,y,z) = \int_0^\infty dk \int_0^\infty d\alpha \int_0^\infty d\mu \times \\
&\quad \times a_2 b_2 c_2 d_1\, sen\,(\mu x)\, sen\,(\beta y)\, sen\,(\gamma z)\,, \\
\frac{\partial u(x,y,z,0)}{\partial t} &= g(x,y,z) = \int_0^\infty dk \int_0^\infty d\alpha \int_0^\infty d\mu \times \\
&\quad \times a_2 b_2 c_2 d_2 k v\, sen\,(\mu x)\, sen\,(\beta y)\, sen\,(\gamma z)\,.
\end{aligned}
$$

Considerando primeiro a integral em μ*, escrevemos as equações acima como,*

$$
\begin{aligned}
f(x,y,z) &= \frac{2}{\pi} \int_0^\infty d\mu\, sen\,(\mu x) \int_0^\infty dk \int_0^\infty d\alpha \times \\
&\quad \times \frac{\pi}{2} a_2 b_2 c_2 d_1\, sen\,(\beta y)\, sen\,(\gamma z)\,, \\
g(x,y,z) &= \frac{2}{\pi} \int_0^\infty d\mu\, sen\,(\mu x) \int_0^\infty dk \int_0^\infty d\alpha \times \\
&\quad \times \frac{\pi}{2} a_2 b_2 c_2 d_2 k v\, sen\,(\beta y)\, sen\,(\gamma z)\,,
\end{aligned}
$$

portanto,

$$
\begin{aligned}
&\int_0^\infty dk \int_0^\infty d\alpha \frac{\pi}{2} a_2 b_2 c_2 d_1\, sen\,(\beta y)\, sen\,(\gamma z) = \\
&= \int_0^\infty f(\xi, y, z)\, sen\,(\mu \xi) d\xi\,.
\end{aligned}
$$

Como,

$$\beta^2 = \alpha^2 - \mu^2 > 0\,, \quad \gamma^2 = k^2 - \alpha^2 > 0\,,$$

as integrais em k e α não são triviais.

6. Determine $u(t, x, y, z)$ para os seguintes problemas:

(a)

$$\frac{\partial^2 u}{\partial t^2} = v^2 \nabla^2 u + f(x, y, z)\,,$$

(b)

$$\frac{\partial^2 u}{\partial t^2} = v^2 \nabla^2 u + \alpha^2\,,$$

com α constante,

(c)

$$\frac{\partial^2 u}{\partial t^2} = v^2 \nabla^2 + \alpha^2 u\,,$$

com α constante e,

$$\begin{aligned}
&0 \leq x \leq a\,, \quad 0 \leq y \leq b\,, \quad 0 \leq z \leq c\,,\\
&u(t, 0, y, z) = \mu_1(y, z)\,, \quad u(t, a, y, z) = \mu_2(y, z)\,,\\
&u(t, x, 0, z) = \nu_1(x, z)\,, \quad u(t, x, b, z) = \nu_2(x, z)\,,\\
&u(t, x, y, 0) = \eta_1(x, y)\,, \quad u(t, x, y, c) = \eta_2(x, y)\,,\\
&u(0, x, y, z) = \varphi(x, y, z)\,, \quad u_t(0, x, y, z) = \psi(x, y, z)\,.
\end{aligned}$$

7. Determine $u(t, x, y, z)$ para os seguintes problemas:

(a)

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2} + f(t, x, y, z)\,,$$

(b)

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2} + \alpha^2\,,$$

com α constante,

(c)

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2} + \alpha^2 u\,,$$

com α constante e,

$$
\begin{aligned}
&0 \le x \le a\,, \quad 0 \le y \le b\,, \quad 0 \le z \le c\,, \\
&u(t,0,y,z) = \mu_1(t,y,z)\,, \quad u(t,a,y,z) = \mu_2(t,y,z)\,, \\
&u(t,x,0,z) = \nu_1(t,x,z)\,, \quad u(t,x,b,z) = \nu_2(t,x,z)\,, \\
&u(t,x,y,0) = \eta_1(t,x,y)\,, \quad u(t,x,y,c) = \eta_2(t,x,y)\,, \\
&u(0,x,y,z) = \varphi(x,y,z)\,, \quad u_t(0,x,y,z) = \psi(x,y,z)\,.
\end{aligned}
$$

4 Apêndice

1. Série de Fourier

$$
\begin{aligned}
f(x) &= \frac{a_0}{2} + \sum_{m=1}^{\infty} [a_m \cos(m\pi x/L) + b_m \,\mathrm{sen}\,(m\pi x/L)]\,, \\
\frac{a_0}{2} &= \frac{1}{2L}\int_{-L}^{L} f(x)\,dx\,, \\
a_m &= \frac{1}{L}\int_{-L}^{L} f(x)\cos(m\pi x/L)\,dx\,, \\
b_m &= \frac{1}{L}\int_{-L}^{L} f(x)\,\mathrm{sen}\,(m\pi x/L)\,dx\,, \quad m = 0,1,2,\ldots
\end{aligned}
$$

2. Série de Fourier de senos

$$
f(z) = \sum_{j=1} b_j \,\mathrm{sen}\,(j\pi z/L)\,,
$$

$$
b_j = \frac{2}{L}\int_0^L f(x)\,\mathrm{sen}\,(j\pi x/L)\,dx\,.
$$

3. Série de Fourier de cosenos

$$
f(z) = \frac{a_0}{2} + \sum_{j=1} a_j \cos(j\pi z/L)\,,
$$

$$
a_j = \frac{2}{L}\int_0^L f(x)\cos(j\pi x/L)\,dx\,.
$$

4. Série de Fourier dupla

$$
f(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}\,\mathrm{sen}\,\frac{m\pi x}{L_1}\,\mathrm{sen}\,\frac{n\pi y}{L_2}\,,
$$

$$B_{mn} = \frac{4}{L_1 L_2} \int_0^{L_1} \int_0^{L_2} f(x,y) \operatorname{sen} \frac{m\pi x}{L_1} \operatorname{sen} \frac{n\pi y}{L_2} dxdy \, .$$

5. Série de Fourier tripla

$$f(x,y,z) = \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{lmn} \operatorname{sen} \frac{l\pi x}{L_1} \operatorname{sen} \frac{m\pi y}{L_2} \operatorname{sen} \frac{n\pi z}{L_3} \, ,$$

$$B_{lmn} = \frac{8}{L_1 L_2 L_3} \int_0^{L_1} \int_0^{L_2} \int_0^{L_3} f(x,y,z) \operatorname{sen} \frac{l\pi x}{L_1} \operatorname{sen} \frac{m\pi y}{L_2} \operatorname{sen} \frac{n\pi z}{L_3} dxdydz \, .$$

6.

$$\int_0^a \operatorname{sen}(i\pi\xi/a) d\xi = \begin{cases} \dfrac{2a}{i\pi} \, , & i \text{ ímpar} \, , \\ 0 \, , & i \text{ par} \, . \end{cases}$$

7.

$$\int \cos ax \, \cos bx \, dx = \frac{1}{2} \frac{sen\,(a+b)x}{a+b} + \frac{1}{2} \frac{sen\,(a-b)x}{a-b} \, ,$$

$$\begin{aligned} \int_{-L}^{L} \cos(i\pi x/L) \, \cos(j\pi x/L) \, dx &= \left. \frac{1}{2} \frac{sen\,[(i+j)\pi x/L]}{(i+j)\pi/L} + \frac{1}{2} \frac{sen\,[(i-j)\pi x/L]}{(i-j)\pi/L} \right|_{-L}^{L} \, , \\ &= \begin{cases} 0 \, , & i \neq j \, , \\ L \, , & i = j \, , \end{cases} \end{aligned}$$

$$\begin{aligned} \int_{0}^{L} \cos(i\pi x/L) \, \cos(j\pi x/L) \, dx &= \left. \frac{1}{2} \frac{sen\,[(i+j)\pi x/L]}{(i+j)\pi/L} + \frac{1}{2} \frac{sen\,[(i-j)\pi x/L]}{(i-j)\pi/L} \right|_{0}^{L} \, , \\ &= \begin{cases} 0 \, , & i \neq j \, , \\ L/2 \, , & i = j \, . \end{cases} \end{aligned}$$

8.

$$\int \operatorname{sen} ax \operatorname{sen} bx \, dx = \frac{1}{2} \frac{sen\,(a-b)x}{a-b} - \frac{1}{2} \frac{sen\,(a+b)x}{a+b} \, ,$$

$$\int_{-L}^{L} \text{sen } (i\pi x/L) \text{ sen } (j\pi x/L)\, dx = \frac{1}{2}\frac{sen\,[(i-j)\pi x/L]}{(i-j)\pi/L} - \frac{1}{2}\frac{sen\,[(i+j)\pi x/L]}{(i+j)\pi/L}\Bigg|_{-L}^{L},$$
$$= \begin{cases} 0\,, & i \neq j\,, \\ L\,, & i = j\,, \end{cases}$$

$$\int_{0}^{L} \text{sen } (i\pi x/L) \text{ sen } (j\pi x/L)\, dx = \frac{1}{2}\frac{sen\,[(i-j)\pi x/L]}{(i-j)\pi/L} - \frac{1}{2}\frac{sen\,[(i+j)\pi x/L]}{(i+j)\pi/L}\Bigg|_{0}^{L},$$
$$= \begin{cases} 0\,, & i \neq j\,, \\ L/2\,, & i = j\,. \end{cases}$$

9.

$$\int \text{sen } ax \cos bx\, dx = -\frac{1}{2}\frac{\cos(a+b)x}{a+b} - \frac{1}{2}\frac{\cos(a-b)x}{a-b}\,.$$

$$\int_{-L}^{L} \text{sen } (i\pi x/L) \cos(j\pi x/L)\, dx = -\frac{1}{2}\frac{\cos[(i+j)\pi x/L]}{(i+j)\pi/L} - \frac{1}{2}\frac{\cos\,[(i-j)\pi x/L]}{(i-j)\pi/L}\Bigg|_{-L}^{L},$$
$$= 0\,,$$

$$\int_{0}^{L} \text{sen } (i\pi x/L) \cos(j\pi x/L)\, dx = -\frac{1}{2}\frac{\cos[(i+j)\pi x/L]}{(i+j)\pi/L} - \frac{1}{2}\frac{\cos\,[(i-j)\pi x/L]}{(i-j)\pi/L}\Bigg|_{0}^{L},$$
$$= \begin{cases} \dfrac{1-(-1)^{i+j}}{2(i+j)\pi/L} + \dfrac{1-(-1)^{i+j}}{2(i-j)\pi/L}\,, & i \neq j\,, \\ 0\,, & i = j\,. \end{cases}$$

10.

$$\int_{0}^{L} x \,\text{sen } (i\pi x/L)\, dx = \begin{cases} +\dfrac{L^2}{i\pi}\,, & i \text{ ímpar}\,, \\ -\dfrac{L^2}{i\pi}\,, & i \text{ par}\,, \end{cases}$$
$$= \frac{L^2}{i\pi}(-1)^{i+1}\,,$$

$$\int x \operatorname{sen}\, ax\, dx = \frac{\operatorname{sen}\, ax}{a^2} - \frac{x \cos ax}{a}\,.$$

11.

$$x = \frac{2L}{\pi}\sum_{i=1} \frac{\operatorname{sen}\,(i\pi x/L)}{i}(-1)^{i+1}\,, \quad 0 < x < L\,.$$

12.

$$1 = \frac{4}{\pi}\sum_{i=1} \frac{\operatorname{sen}\,[(2i-1)\pi x/L]}{2i-1}\,, \quad 0 < x < L\,.$$

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