8 - A equação do calor em coordenadas esféricas

Procedemos aqui da mesma forma que no caso da equação do calor em coordenadas cilíndricas. A equação da condução do calor é,

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u \,. \tag{1}$$

Escrevendo a solução u na forma,

$$u(t, r, \theta, \varphi) = F(r, \theta, \varphi) T(t) \,. \tag{2}$$

temos como antes,

$$T(t) = A \exp(-\kappa \lambda^2 t) \,. \tag{3}$$

A equação para F é,

$$\nabla^2 F + \lambda^2 F = 0 \,. \tag{4}$$

Consideramos agora a equação do calor com todas as variáveis explicitamente. Em coordenadas esféricas, a equação (4) fica,

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial F}{\partial r} \right) + \frac{1}{r^2 \mathrm{sen}\, \theta} \frac{\partial}{\partial \theta} \left(\mathrm{sen}\, \theta \frac{\partial F}{\partial \theta} \right) + \frac{1}{r^2 \mathrm{sen}^2\, \theta} \frac{\partial^2 F}{\partial \varphi^2} + \lambda^2 F = 0 \,. \tag{5}$$

Consideramos agora os diversos casos possíveis.

1 Considerando $F = F(r)$

A equação (5) fica,

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dF}{dr} \right) + \lambda^2 F = 0 \,, \tag{6}$$

ou,

$$r^2 F'' + 2 r F' + (\lambda r)^2 F = 0 \,, \tag{7}$$

com solução [6,7],

$$F(r) = r^{-1/2} [a_1 J_{1/2}(\lambda r) + a_2 Y_{1/2}(\lambda r)] \,. \tag{8}$$

A solução geral é então,

$$u(t,r) = T(t)F(r) = \exp(-\lambda^2\kappa t)r^{-1/2}[a_1 J_{1/2}(\lambda r) + a_2 Y_{1/2}(\lambda r)]\,, \tag{9}$$

em que fizemos $A = 1$. Usando o princípio da superposição a solução geral é,

$$\begin{aligned} u(t,r) = {} & c_0 + d_0 u_0(r) \\ & + \sum_j \exp(-\lambda_j^2\kappa t)r^{-1/2}[a_{1j} J_{1/2}(\lambda_j r) + a_{2j} Y_{1/2}(\lambda_j r)]\,, \end{aligned} \tag{10}$$

em que c_0, d_0 são constantes e u_0 é uma solução da equação de Laplace. Os dois primeiros termos correspondem à solução estacionária, obtida fazendo o limite $t \to \infty$. Se escolhemos $c_0 = 0$ e $d_0 = 1$ a solução é,

$$u(t,r) = u_0(r) + \sum_j \exp(-\lambda_j^2\kappa t)r^{-1/2}[a_{1j} J_{1/2}(\lambda_j r) + a_{2j} Y_{1/2}(\lambda_j r)]\,. \tag{11}$$

2 Problemas

1. Considere uma esfera de raio a. Calcule $u(t,r)$ sendo $u(t,a) = u_a$ e $u(0,r) = f(r)$.

Escrevemos a solução finita como,

$$u(t,r) = u_0(r) + \sum_j a_{1j} \exp(-\lambda_j^2\kappa t)r^{-1/2} J_{1/2}(\lambda_j r)\,.$$

A solução estacionária é $u_0(r)$, a solução da equação de Laplace, correspondente ao limite $t \to \infty$. Em $r = a$,

$$u(t,a) = u_a = u_0(a) + \sum_j a_{1j} \exp(-\lambda_j^2\kappa t)r^{-1/2} J_{1/2}(\lambda_j a)\,.$$

Satisfazemos a equação acima escolhendo,

$$J_{1/2}(\lambda_j a) = 0\,, \quad j = 1, 2, \ldots$$

A condição acima define os valores possíveis de λ_j. Portanto,

$$u_a = u_0(a)\,,$$

e a solução estacionária é constante, $u_0(r) = u_0(a) = u_a$. A condição inicial nos dá,

$$u(0,r) = f(r) = u_a + \sum_j a_{1j} r^{-1/2} J_{1/2}(\lambda_j r)\,,$$

ou,

$$f(r) - u_a = \sum_j a_{1j} r^{-1/2} J_{1/2}(\lambda_j r)\,.$$

Temos assim uma expansão em série de funções de Bessel. Multiplicando a equação acima por $r^{3/2} J_{1/2}(\lambda_k r)$ e integrando em r,

$$\int_0^a [f(r) - u_a] r^{3/2} J_{1/2}(\lambda_k r) dr = \sum_j a_{1j} \int_0^a r J_{1/2}(\lambda_j r) J_{1/2}(\lambda_k r) dr\,.$$

Usando a ortogonalidade das funções de Bessel,

$$\int_0^a r J_{1/2}(\lambda_j r) J_{1/2}(\lambda_k r) dr = \begin{cases} 0\,, & j \neq k\,, \\ \dfrac{a^2}{2} [J'_{1/2}(\lambda_j a)]^2\,, & j = k\,, \end{cases}$$

temos,

$$\int_0^a [f(r) - u_a] r^{3/2} J_{1/2}(\lambda_k r) dr = a_{1k} \frac{a^2}{2} [J'_{1/2}(\lambda_k a)]^2\,.$$

Portanto,

$$a_{1k} = \frac{2}{a^2 [J'_{1/2}(\lambda_k a)]^2} \int_0^a [f(r) - u_a] r^{3/2} J_{1/2}(\lambda_k r) dr\,.$$

A solução é assim,

$$\begin{aligned} u(t,r) = u_a + \sum_j \exp(-\lambda_j^2 \kappa t) r^{-1/2} J_{1/2}(\lambda_j r) \times \\ \times \frac{2}{a^2 [J'_{1/2}(\lambda_j a)]^2} \int_0^a [f(r) - u_a] r^{3/2} J_{1/2}(\lambda_j r) dr\,. \end{aligned}$$

(b) Podemos simplificar a expressão acima substituindo,

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\, sen\, x\,.$$

Portanto,

$$u(t,r) = u_a + \sum_j \exp(-\lambda_j^2 \kappa t) \frac{1}{r} \sqrt{\frac{2}{\pi \lambda_j}} sen\,(\lambda_j r) \times$$
$$\times \frac{2}{a^2 [J'_{1/2}(\lambda_j a)]^2} \sqrt{\frac{2}{\pi \lambda_j}} \int_0^a [f(r) - u_a] r sen\,(\lambda_j r) dr\,.$$

Os valores de λ *são dados por,*

$$J_{1/2}(\lambda_j a) = \sqrt{\frac{2}{\pi \lambda_j a}} sen\,(\lambda_j a) = 0\,, \quad j = 1, 2, \ldots$$

ou,

$$\lambda_j a = j\pi\,.$$

Assim,

$$u(t,r) = u_a + \sum_j \exp[-(j\pi/a)^2 \kappa t] \frac{1}{r} \frac{2a}{\pi^2 j} sen\,(j\pi r/a) \times$$
$$\times \frac{2}{a^2 [J'_{1/2}(j\pi)]^2} \int_0^a [f(r) - u_a] r sen\,(j\pi r/a) dr\,.$$

Substituindo $J'_{1/2}(\pi j) = (-1)^j \sqrt{2/\pi^2 j}$ *temos,*

$$u(t,r) = u_a + \frac{2a}{r} \sum_j \exp[-(j\pi/a)^2 \kappa t] sen\,(j\pi r/a) \times$$
$$\times \frac{1}{a^2} \int_0^a [f(r) - u_a] r sen\,(j\pi r/a) dr\,.$$

(c) Se $f(r) = f_0$ *constante temos,*

$$u(t,r) = u_a + \frac{2a}{r} \sum_j \exp[-(j\pi/a)^2 \kappa t] sen\,(j\pi r/a) \times$$
$$\times \frac{1}{a^2} (f_0 - u_a) \int_0^a r sen\,(j\pi r/a) dr\,.$$

Usando,

$$\int x sen\, bx\, dx = \frac{sen\, bx}{b^2} - \frac{x \cos bx}{b}\,,$$

temos,

$$u(t,r) = u_a + \frac{2a}{r\pi}(f_0 - u_a) \times$$
$$\times \sum_j \frac{(-1)^{j+1}}{j} \exp[-(j\pi/a)^2 \kappa t] sen\,(j\pi r/a)\,.$$

Se $f_0 = 0$ *(Churchill [12], probl. 6.1),*

$$u(t,r) = u_a + \frac{2au_a}{r\pi} \times$$
$$\times \sum_j \frac{(-1)^j}{j} \exp[-(j\pi/a)^2 \kappa t] sen\,(j\pi r/a)\,.$$

(d) Consideremos uma esfera de ferro com raio $a = 20$ *cm e* $\kappa = 0,15$ *(unidades c.g.s.). Sendo* $u_a = 0^o C$ *e* $f(r) = f_0 = 100^o C$*, constante, calcule a temperatura no centro da esfera após 10 min (Churchill [12], probl. 6.3).*

Temos, em $r = 0$*,*

$$u(t,r) = u_a + 2(f_0 - u_a) \times$$
$$\times \sum_j (-1)^{j+1} \exp[-(j\pi/a)^2 \kappa t]\,.$$

A soma acima converge rapidamente para 0,1084, e obtemos $T = 21,68^o C$*.*

2. Considere a região $a \leq r \leq b$. Calcule $u(t,r)$ sendo $u(t,a) = u_a$, $u(t,b) = u_b$ e $u(0,r) = f(r)$.

(a) Escrevemos a solução como,

$$u(t,r) = u_0(r) + \sum_j \exp(-\lambda_j^2 \kappa t) r^{-1/2} [a_{1j} J_{1/2}(\lambda_j r) + a_{2j} Y_{1/2}(\lambda_j r)]\,.$$

As condições de contorno em $r = a$ *e* $r = b$ *nos dão,*

$$u(t,a) = u_a = u_0(a) + \sum_j \exp(-\lambda_j^2 \kappa t) a^{-1/2} [a_{1j} J_{1/2}(\lambda_j a) + a_{2j} Y_{1/2}(\lambda_j a)]\,,$$
$$u(t,b) = u_b = u_0(b) + \sum_j \exp(-\lambda_j^2 \kappa t) b^{-1/2} [a_{1j} J_{1/2}(\lambda_j b) + a_{2j} Y_{1/2}(\lambda_j b)]\,.$$

Satisfazemos as equações acima escolhendo,

$$a_{1j} J_{1/2}(\lambda_j a) + a_{2j} Y_{1/2}(\lambda_j a) = 0\,,$$
$$a_{1j} J_{1/2}(\lambda_j b) + a_{2j} Y_{1/2}(\lambda_j b) = 0\,.$$

Devemos ter portanto,

$$J_{1/2}(\lambda_j a)Y_{1/2}(\lambda_j b) - Y_{1/2}(\lambda_j a)J_{1/2}(\lambda_j b) = 0\,.$$

A equação acima determina os valores possíveis de λ. Também temos,

$$u_0(a) = u_a\,,$$
$$u_0(b) = u_b\,.$$

Usando a definição,

$$u_{1/2}(\lambda_j r) \equiv Y_{1/2}(\lambda_j a)J_{1/2}(\lambda_j r) - J_{1/2}(\lambda_j a)Y_{1/2}(\lambda_j r)\,,$$

a condição que define os valores de λ fica,

$$u_{1/2}(\lambda_j b) = 0\,.$$

Também podemos escrever,

$$a_{2j} = -a_{1j}\frac{J_{1/2}(\lambda_j a)}{Y_{1/2}(\lambda_j a)}\,.$$

Resta determinar a_{1j}. A condição inicial nos dá,

$$u(0,r) = f(r) = u_0(r) + \sum_j r^{-1/2}[a_{1j}J_{1/2}(\lambda_j r) + a_{2j}Y_{1/2}(\lambda_j r)]\,.$$

Substituindo a_{2j},

$$f(r) = u_0(r) + \sum_j a_{1j} r^{-1/2} u_{1/2}(\lambda_j r)\frac{1}{Y_{1/2}(\lambda_j a)}\,,$$

ou,

$$f(r) - u_0(r) = \sum_j a_{1j} r^{-1/2} u_{1/2}(\lambda_j r)\frac{1}{Y_{1/2}(\lambda_j a)}\,,$$

Multiplicando por $r^{3/2}u_{1/2}(\lambda_k r)$ e integrando em r,

$$\int_a^b [f(r) - u_0(r)] r^{3/2} u_{1/2}(\lambda_k r) dr =$$
$$= \sum_j a_{1j} \int_a^b r u_{1/2}(\lambda_k r) u_{1/2}(\lambda_j r) \frac{1}{Y_{1/2}(\lambda_j a)} dr \,,$$
$$= a_{1k} \int_a^b r u_{1/2}^2(\lambda_k r) \frac{1}{Y_{1/2}(\lambda_k a)} dr \,,$$

em que usamos a relação de ortogonalidade entre as funções $u_{1/2}(\lambda_k r)$ *(ver apêndice). Os coeficientes* a_{1j} *são então,*

$$a_{1j} = Y_{1/2}(\lambda_j a) \frac{\displaystyle\int_a^b [f(r) - u_0(r)] r^{3/2} u_{1/2}(\lambda_j r) dr}{\displaystyle\int_a^b r u_{1/2}^2(\lambda_j r) dr} \,.$$

A solução é portanto,

$$u(t,r) = u_0(r) + \sum_j \exp(-\lambda_j^2 \kappa t) r^{-1/2} u_{1/2}(\lambda_j r) \times$$
$$\times \frac{1}{\displaystyle\int_a^b r u_{1/2}^2(\lambda_j r) dr} \times$$
$$\times \int_a^b [f(r) - u_0(r)] r^{3/2} u_{1/2}(\lambda_j r) dr \,.$$

(b) Vamos simplificar a expressão acima substituindo,

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} sen\, x \,, \quad Y_{1/2}(x) = -\sqrt{\frac{2}{\pi x}} \cos\, x \,.$$

Temos,

$$u_{1/2}(\lambda_j r) = Y_{1/2}(\lambda_j a) J_{1/2}(\lambda_j r) - J_{1/2}(\lambda_j a) Y_{1/2}(\lambda_j r) \,,$$
$$= -\frac{1}{\sqrt{ar}} \frac{2}{\pi \lambda_j} \cos(\lambda_j a) sen(\lambda_j r) \,,$$
$$+ \frac{1}{\sqrt{ar}} \frac{2}{\pi \lambda_j} sen(\lambda_j a) \cos(\lambda_j r) \,,$$
$$= \frac{1}{\sqrt{ar}} \frac{2}{\pi \lambda_j} sen\, \lambda_j (a - r) \,,$$
$$= -\frac{1}{\sqrt{ar}} \frac{2}{\pi \lambda_j} sen\, \lambda_j (r - a) \,.$$

A condição que determina λ *é então,*

$$u_{1/2}(\lambda_j b) = -\frac{1}{\sqrt{ar}}\frac{2}{\pi\lambda_j} sen\, \lambda_j(b-a) = 0\,.$$

Os valores possíveis de λ são assim,

$$\lambda_j(b-a) = j\pi\,, \quad j = 1, 2, \ldots$$

ou,

$$\lambda_j = \frac{j\pi}{(b-a)}\,, \quad j = 1, 2, \ldots$$

Temos,

$$\begin{aligned} \int_a^b r u_{1/2}^2(\lambda_j r) dr &= \frac{4}{a\pi^2\lambda_j^2}\int_a^b sen^2\, \lambda_j(r-a) dr\,, \\ &= \frac{2(b-a)}{a\pi^2\lambda_j^2}\,, \end{aligned}$$

em que usamos,

$$\int sen^2 cx\, dx = \frac{x}{2} - \frac{sen\, 2cx}{4c}\,.$$

A solução é então,

$$\begin{aligned} u(t,r) = u_0(r) + \frac{1}{r}\sum_j \exp[-j^2\pi^2\kappa t/(b-a)^2] sen\, \frac{j\pi(r-a)}{(b-a)} \times \\ \times \frac{2}{b-a}\int_a^b [f(r) - u_0(r)] r\, sen\, \frac{j\pi(r-a)}{(b-a)} dr\,. \end{aligned}$$

Se $u_a = u_b = 0$ temos $u_0(r) = 0$ (Churchill [12], probl. 6.5).

3. Calcule $u(t,r)$ em $0 \le r \le a$ com as condições,

$$\begin{aligned} u(a,t) &= \mu(t)\,, \\ u(r,0) &= \varphi(r)\,. \end{aligned}$$

4. Calcule $u(t,r)$ em $a \le r \le b$ com as condições,

$$\begin{aligned} u(a,t) &= \mu_1(t)\,, \\ u(b,t) &= \mu_2(t)\,, \\ u(r,0) &= \varphi(r)\,. \end{aligned}$$

5. Calcule a solução $u(t,r)$ da equação,

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + \alpha u \, ,$$

em $0 \leq r \leq a$ com as condições,

$$u(a,t) = \mu(t) \, ,$$
$$u(r,0) = \varphi(r) \, .$$

6. Calcule a solução $u(t,r)$ da equação,

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + \alpha^2 \, ,$$

em $0 \leq r \leq a$ com as condições,

$$u(a,t) = \mu(t) \, ,$$
$$u(r,0) = \varphi(r) \, .$$

7. Calcule a solução $u(t,r)$ da equação,

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + f(t,r) \, ,$$

em $0 \leq r \leq a$ com as condições,

$$u(a,t) = \mu(t) \, ,$$
$$u(r,0) = \varphi(r) \, .$$

8. Calcule a solução $u(t,r)$ da equação,

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + \alpha u \, ,$$

em $a \leq r \leq b$ com as condições,

$$u(a,t) = \mu_1(t) \, ,$$
$$u(b,t) = \mu_2(t) \, ,$$
$$u(r,0) = \varphi(r) \, .$$

9. Calcule a solução $u(t,r)$ da equação,

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + \alpha^2 \,,$$

em $a \leq r \leq b$ com as condições,

$$\begin{aligned} &u(a,t) = \mu_1(t)\,, \\ &u(b,t) = \mu_2(t)\,, \\ &u(r,0) = \varphi(r)\,. \end{aligned}$$

10. Calcule a solução $u(t,r)$ da equação,

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + f(t,r)\,,$$

em $a \leq r \leq b$ com as condições,

$$\begin{aligned} &u(a,t) = \mu_1(t)\,, \\ &u(b,t) = \mu_{@}(t)\,, \\ &u(r,0) = \varphi(r)\,. \end{aligned}$$

3 Considerando $F(r,\theta)$

Agora a equação para F é,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial F}{\partial r}\right) + \frac{1}{r^2 \mathrm{sen}\,\theta}\frac{\partial}{\partial \theta}\left(\mathrm{sen}\,\theta\frac{\partial F}{\partial \theta}\right) + \lambda^2 F = 0\,. \tag{12}$$

Substituindo,

$$F(r,\theta) = R(r)\Theta(\theta)\,, \tag{13}$$

temos,

$$\frac{1}{R}(2rR' + r^2R'') + \lambda^2 r^2 = -\frac{1}{\mathrm{sen}\,\theta\Theta}(\mathrm{sen}\,\theta\Theta')' \equiv -\mu^2\,.$$

Da equação acima obtemos,

$$\begin{aligned} &r^2R'' + 2rR' + (\lambda^2 r^2 + \mu^2)R = 0\,, \\ &(\mathrm{sen}\,\theta\Theta')' - \mu^2 \mathrm{sen}\,\theta\Theta = 0\,. \end{aligned}$$

A solução para R é [6,7],

$$R(r) = r^{-1/2}[c_1 J_{n+1/2}(\lambda r) + c_2 Y_{n+1/2}(\lambda r)]\,, \tag{14}$$

com,

$$n(n+1) = -\mu^2\,. \tag{15}$$

A equação para Θ fica então,

$$(\operatorname{sen}\theta\Theta')' + n(n+1)\operatorname{sen}\theta\Theta = 0\,, \tag{16}$$

que é a equação diferencial de Legendre, com solução,

$$\Theta(\theta) = b_1 P_n(\cos\theta) + b_2 Q_n(\cos\theta)\,. \tag{17}$$

A solução geral da equação do calor é então,

$$\begin{aligned} u(t,r,\theta) = {} & c_0 + d_0 u_0(r,\theta) + A\exp(-\kappa\lambda^2 t) r^{-1/2} \times \\ & \times[c_1 J_{n+1/2}(\lambda r) + c_2 Y_{n+1/2}(\lambda r)] \times \\ & \times[b_1 P_n(\cos\theta) + b_2 Q_n(\cos\theta)]\,, \end{aligned}$$

ou, usando o princípio da superposição e fazendo $A = 1$,

$$u(t,r,\theta) = c_0 + d_0 u_0(r,\theta) + \sum_{\lambda n} \exp(-\kappa\lambda^2 t) r^{-1/2} \times \tag{18}$$

$$\times[c_1 J_{n+1/2}(\lambda r) + c_2 Y_{n+1/2}(\lambda r)] \times$$

$$\times[b_1 P_n(\cos\theta) + b_2 Q_n(\cos\theta)]\,, \tag{19}$$

em que u_0 é solução da equação de Laplace, como antes. Escolhendo $c_0 = 0$ e $d_0 = 1$ temos,

$$u(t,r,\theta) = u_0(r,\theta) + \sum_{\lambda n} \exp(-\kappa\lambda^2 t) r^{-1/2} \times \tag{20}$$

$$\times[c_1 J_{n+1/2}(\lambda r) + c_2 Y_{n+1/2}(\lambda r)] \times$$

$$\times[b_1 P_n(\cos\theta) + b_2 Q_n(\cos\theta)]\,, \tag{21}$$

4 Problemas

1. Considere uma esfera de raio a. Calcule a temperatura $u(t,r,\theta)$ sendo $u(t,a,\theta) = f(\theta)$ e $u(0,r,\theta) = g(r,\theta)$.

(a) A solução finita é,

$$u(t,r,\theta) = u_0(r,\theta) + \sum_{\lambda n} b_1 \exp(-\kappa\lambda^2 t) r^{-1/2} J_{n+1/2}(\lambda r) P_n(\cos\theta) \,.$$

A condição de contorno em $r = a$ nos dá,

$$\begin{aligned} u(t,a,\theta) = f(\theta) &= u_0(a,\theta) \\ &+ \sum_{\lambda n} b_1 \exp(-\kappa\lambda^2 t) a^{-1/2} J_{n+1/2}(\lambda a) P_n(\cos\theta) \,. \end{aligned}$$

Satisfazemos a condição acima escolhendo,

$$J_{n+1/2}(\lambda_{nj} a) = 0 \,, \quad j = 1, 2, \ldots$$

A equação acima determina os valores possíveis de λ_j. Temos também,

$$f(\theta) = u_0(a,\theta) \,.$$

Considerando agora a condição inicial temos,

$$\begin{aligned} u(0,r,\theta) = g(r,\theta) &= u_0(r,\theta) \\ &+ \sum_{nj} b_1 r^{-1/2} J_{n+1/2}(\lambda_{nj} r) P_n(\cos\theta) \,, \end{aligned}$$

ou,

$$g(r,\theta) - u_0(r,\theta) = \sum_{nj} b_1 r^{-1/2} J_{n+1/2}(\lambda_{nj} r) P_n(\cos\theta) \,.$$

Temos uma série de polinômios de Legendre, logo,

$$\begin{aligned} &\sum_j b_1 r^{-1/2} J_{n+1/2}(\lambda_{nj} r) = \\ &= \frac{2n+1}{2} \int_0^\pi [g(r,\theta) - u_0(r,\theta)] P_n(\cos\theta) sen\,\theta d\theta \,. \end{aligned}$$

Temos agora uma série de funções de Bessel. Multiplicando a equação acima por $r^{3/2} J_{n+1/2}(\lambda_{nk} r)$ e integrando em r,

$$
\begin{aligned}
&\sum_j b_1 \int_0^a r J_{n+1/2}(\lambda_{nj} r) J_{n+1/2}(\lambda_{nk} r) dr = \\
&= \int_0^a dr\, r^{3/2} J_{n+1/2}(\lambda_{nk} r) \times \\
&\times \frac{2n+1}{2} \int_0^\pi [g(r,\theta) - u_0(r,\theta)] P_n(\cos\theta) sen\,\theta d\theta\,.
\end{aligned}
$$

Usando a condição de ortogonalidade,

$$
\int_0^a r J_{n+1/2}(\lambda_{nj} r) J_{n+1/2}(\lambda_{nk} r) dr = \begin{cases} 0\,, & j \neq k\,, \\ \dfrac{a^2}{2} [J'_{n+1/2}(\lambda_{nj} a)]^2\,, & j = k\,, \end{cases}
$$

temos,

$$
\begin{aligned}
&b_1 \frac{a^2}{2} [J'_{n+1/2}(\lambda_{nk} a)]^2 = \\
&= \int_0^a dr\, r^{3/2} J_{n+1/2}(\lambda_{nk} r) \times \\
&\times \frac{2n+1}{2} \int_0^\pi [g(r,\theta) - u_0(r,\theta)] P_n(\cos\theta) sen\,\theta d\theta\,.
\end{aligned}
$$

Portanto,

$$
\begin{aligned}
&b_1 = \frac{2}{a^2 [J'_{n+1/2}(\lambda_{nk} a)]^2} \times \\
&\times \int_0^a dr\, r^{3/2} J_{n+1/2}(\lambda_{nk} r) \times \\
&\times \frac{2n+1}{2} \int_0^\pi [g(r,\theta) - u_0(r,\theta)] P_n(\cos\theta) sen\,\theta d\theta\,.
\end{aligned}
$$

A solução é portanto,

$$
\begin{aligned}
u(t,r,\theta) = {} & u_0(r,\theta) \\
& + \sum_{nj} \exp(-\kappa \lambda_{nj}^2 t) r^{-1/2} J_{n+1/2}(\lambda_{nj} r) P_n(\cos\theta) \times \\
& \times \frac{2}{a^2 [J'_{n+1/2}(\lambda_{nj} a)]^2} \times \\
& \times \int_0^a dr\, r^{3/2} J_{n+1/2}(\lambda_{nj} r) \times \\
& \times \frac{2n+1}{2} \int_0^\pi [g(r,\theta) - u_0(r,\theta)] P_n(\cos\theta) sen\,\theta d\theta\,.
\end{aligned}
$$

(b) Se $f = f_0$ *constante,* $u_0(r,\theta) = f_0$*, logo, se* $g = g(r)$*,*

$$
\begin{aligned}
u(t,r,\theta) = {} & f_0 + \sum_j \exp(-\kappa\lambda_{0j}^2 t) r^{-1/2} J_{1/2}(\lambda_{0j} r) \times \\
& \times \frac{2}{a^2[J'_{1/2}(\lambda_{0j} a)]^2} \times \\
& \times \int_0^a dr\, r^{3/2} J_{1/2}(\lambda_{0j} r)[g(r) - f_0] \,.
\end{aligned}
$$

2. Considere a região $a \leq r \leq b$. Calcule a temperatura $u(t,r,\theta)$ sendo $u(t,a,\theta) = f(\theta)$, $u(t,b,\theta) = g(\theta)$, $u(0,r,\theta) = h(r,\theta)$.

(a) Temos agora,

$$
\begin{aligned}
u(t,r,\theta) = {} & u_0(r,\theta) + \sum_{\lambda n} \exp(-\kappa\lambda^2 t) r^{-1/2} \times \\
& \times [c_1 J_{n+1/2}(\lambda r) + c_2 Y_{n+1/2}(\lambda r)] P_n(\cos\theta) \,.
\end{aligned}
$$

As condições de contorno em $r = a$ e $r = b$ nos dão as relações,

$$
\begin{aligned}
u(t,a,\theta) = {} & f(\theta) = u_0(a,\theta) + \sum_{\lambda n} \exp(-\kappa\lambda^2 t) a^{-1/2} \times \\
& \times [c_1 J_{n+1/2}(\lambda a) + c_2 Y_{n+1/2}(\lambda a)] P_n(\cos\theta) \,, \\
u(t,b,\theta) = {} & g(\theta) = u_0(b,\theta) + \sum_{\lambda n} \exp(-\kappa\lambda^2 t) b^{-1/2} \times \\
& \times [c_1 J_{n+1/2}(\lambda b) + c_2 Y_{n+1/2}(\lambda b)] P_n(\cos\theta) \,.
\end{aligned}
$$

Satisfazemos as condições acima escolhendo,

$$
\begin{aligned}
& c_1 J_{n+1/2}(\lambda a) + c_2 Y_{n+1/2}(\lambda a) = 0 \,, \\
& c_1 J_{n+1/2}(\lambda b) + c_2 Y_{n+1/2}(\lambda b) = 0 \,.
\end{aligned}
$$

Portanto os valores de λ são dados por,

$$
J_{n+1/2}(\lambda_{nj} a) Y_{n+1/2}(\lambda_{nj} b) - Y_{n+1/2}(\lambda_{nj} a) J_{n+1/2}(\lambda_{nj} b) = 0 \,, \quad j = 1, 2, \ldots
$$

ou,

$$
u_{n+1/2}(\lambda_{nj} b) = 0 \,,
$$

com,

$$u_{n+1/2}(\lambda_{nj}r) \equiv Y_{n+1/2}(\lambda_{nj}a)J_{n+1/2}(\lambda_{nj}r) - J_{n+1/2}(\lambda_{nj}a)Y_{n+1/2}(\lambda_{nj}r)\,.$$

Também temos,

$$\begin{aligned} f(\theta) &= u_0(a,\theta)\,, \\ g(\theta) &= u_0(b,\theta)\,. \end{aligned}$$

Substituindo na solução,

$$c_2 = -c_1 \frac{J_{n+1/2}(\lambda_{nj}a)}{Y_{n+1/2}(\lambda_{nj}a)}\,,$$

temos,

$$\begin{aligned} u(t,r,\theta) = {} & u_0(r,\theta) + \sum_{nj} c_1 \exp(-\kappa\lambda_{nj}^2 t) r^{-1/2} \times \\ & \times u_{n+1/2}(\lambda_{nj}r)\frac{1}{Y_{n+1/2}(\lambda_{nj}a)} P_n(\cos\theta)\,, \end{aligned}$$

A condição inicial nos dá,

$$\begin{aligned} u(0,r,\theta) = {} & h(r,\theta) = u_0(r,\theta) + \sum_{nj} c_1 r^{-1/2} \times \\ & \times u_{n+1/2}(\lambda_{nj}r)\frac{1}{Y_{n+1/2}(\lambda_{nj}a)} P_n(\cos\theta)\,, \end{aligned}$$

ou,

$$\begin{aligned} & h(r,\theta) - u_0(r,\theta) = \\ & = \sum_{nj} c_1 r^{-1/2} u_{n+1/2}(\lambda_{nj}r)\frac{1}{Y_{n+1/2}(\lambda_{nj}a)} P_n(\cos\theta)\,, \end{aligned}$$

Temos uma série de polinômios de Legendre, logo,

$$\begin{aligned} & \sum_j c_1 r^{-1/2}\frac{1}{Y_{n+1/2}(\lambda_{nj}a)} u_{n+1/2}(\lambda_{nj}r) = \\ & = \frac{2n+1}{2}\int_0^\pi [h(r,\theta) - u_0(r,\theta)] P_n(\cos\theta) sen\,\theta\, d\theta\,. \end{aligned}$$

A expressão acima é uma série de funções de Bessel. Multiplicando por $r^{3/2}u_{n+1/2}(\lambda_{nk}r)$ *e integrando em* r*,*

$$\sum_j c_1 \frac{1}{Y_{n+1/2}(\lambda_{nj}a)} \int_a^b dr\, r u_{n+1/2}(\lambda_{nj}r) u_{n+1/2}(\lambda_{nk}r) =$$
$$= \int_a^b dr\, r^{3/2} u_{n+1/2}(\lambda_{nk}r) \times$$
$$\times \frac{2n+1}{2} \int_0^\pi [h(r,\theta) - u_0(r,\theta)] P_n(\cos\theta) sen\,\theta\, d\theta\,.$$

Usando a ortogonalidade das funções de Bessel,

$$\int_a^b r u_{n+1/2}(\lambda_{nj}r) u_{n+1/2}(\lambda_{nk}r) dr = 0\,, \quad j \neq k\,,$$

temos,

$$c_1 \frac{1}{Y_{n+1/2}(\lambda_{nk}a)} \int_a^b dr\, r u_{n+1/2}^2(\lambda_{nk}r) =$$
$$= \int_a^b dr\, r^{3/2} u_{n+1/2}(\lambda_{nk}r) \times$$
$$\times \frac{2n+1}{2} \int_0^\pi [h(r,\theta) - u_0(r,\theta)] P_n(\cos\theta) sen\,\theta\, d\theta\,,$$

ou,

$$c_1 = \frac{Y_{n+1/2}(\lambda_{nk}a)}{\int_a^b dr\, r u_{n+1/2}^2(\lambda_{nk}r)} \int_a^b dr\, r^{3/2} u_{n+1/2}(\lambda_{nk}r) \times$$
$$\times \frac{2n+1}{2} \int_0^\pi [h(r,\theta) - u_0(r,\theta)] P_n(\cos\theta) sen\,\theta\, d\theta\,.$$

A solução é portanto,

$$\begin{aligned} u(t,r,\theta) = {} & u_0(r,\theta) \\ & + \sum_{nj} \exp(-\kappa \lambda_{nj}^2 t) r^{-1/2} \times \\ & \times u_{n+1/2}(\lambda_{nj}r) P_n(\cos\theta) \times \\ & \times \frac{1}{\int_a^b dr\, r u_{n+1/2}^2(\lambda_{nj}r)} \int_a^b dr\, r^{3/2} u_{n+1/2}(\lambda_{nj}r) \times \\ & \times \frac{2n+1}{2} \int_0^\pi [h(r,\theta) - u_0(r,\theta)] P_n(\cos\theta) sen\,\theta\, d\theta\,. \end{aligned}$$

(b) Se $u_0(r,\theta) = u_0(r)$ *e* $h(r,\theta) = h(r)$*,*

$$
\begin{aligned}
u(t,r,\theta) = {} & u_0(r) \\
& + \sum_j \exp(-\kappa\lambda_{0j}^2 t) r^{-1/2} u_{1/2}(\lambda_{0j} r) \times \\
& \times \frac{1}{\int_a^b dr\, r u_{1/2}^2(\lambda_{0j} r)} \int_a^b dr\, r^{3/2} u_{1/2}(\lambda_{0j} r)[h(r) - u_0(r)] \,.
\end{aligned}
$$

3. Calcule $u(t,r,\theta)$ em $0 \leq r \leq a$ com as condições,

$$
\begin{aligned}
u(t,a,\theta) &= \mu(t,\theta)\,, \\
u(0,r,\theta) &= \varphi(r,\theta)\,.
\end{aligned}
$$

4. Calcule $u(t,r,\theta)$ em $a \leq r \leq b$ com as condições,

$$
\begin{aligned}
u(t,a,\theta) &= \mu_1(t,\theta)\,, \\
u(t,b,\theta) &= \mu_2(t,\theta)\,, \\
u(0,r,\theta) &= \varphi(r,\theta)\,.
\end{aligned}
$$

5 Considerando $F(r,\theta,\varphi)$

A equação para F é dada por (5),

$$
\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial F}{\partial r}\right) + \frac{1}{r^2 \operatorname{sen}\theta}\frac{\partial}{\partial\theta}\left(\operatorname{sen}\theta\frac{\partial F}{\partial\theta}\right) + \frac{1}{r^2\operatorname{sen}^2\theta}\frac{\partial^2 F}{\partial\varphi^2} + \lambda^2 F = 0\,.
$$

Substituindo,

$$
F(r,\theta,\varphi) = R(r)\Theta(\theta)\Phi(\varphi)\,,
$$

temos,

$$
\frac{1}{R}(r^2R')' + \frac{1}{\operatorname{sen}\theta\Theta}(\operatorname{sen}\theta\Theta')' + \frac{1}{\operatorname{sen}^2\theta\Phi}\Phi'' + \lambda^2 r^2 = 0\,,
$$

ou,

$$
\frac{1}{R}(r^2R')' + \lambda^2 r^2 = -\frac{1}{\operatorname{sen}\theta\Theta}(\operatorname{sen}\theta\Theta')' - \frac{1}{\operatorname{sen}^2\theta\Phi}\Phi'' \equiv -\mu^2\,. \qquad (22)
$$

A equação para R é assim, como antes,

$$r^2R'' + 2rR' + (\lambda^2r^2 + \mu^2)R = 0\,, \tag{23}$$

com solução,

$$R(r) = r^{-1/2}[c_1J_{n+1/2}(\lambda r) + c_2Y_{n+1/2}(\lambda r)]\,, \tag{24}$$

e,

$$n(n+1) = -\mu^2\,. \tag{25}$$

Para Θ e Φ temos,

$$-\frac{\operatorname{sen}\theta}{\Theta}(\operatorname{sen}\theta\Theta')' + \mu^2\operatorname{sen}^2\theta = \frac{1}{\Phi}\Phi'' \equiv -m^2\,.$$

A equação para Θ é,

$$\operatorname{sen}\theta(\operatorname{sen}\theta\Theta')' + [n(n+1)\operatorname{sen}^2\theta - m^2]\Theta = 0\,, \tag{26}$$

que é a equação diferencial associada de Legendre, com solução,

$$\Theta(\theta) = b_1P_n^m(\cos\theta) + b_2Q_m^m(\cos\theta)\,, \tag{27}$$

em que P_n^m e Q_n^m são funções associadas de Legendre do primeiro e do segundo tipo, respectivamente. A equação para Φ é,

$$\Phi'' + m^2\Phi = 0\,, \tag{28}$$

com solução,

$$\Phi(\varphi) = a_1\cos m\varphi + a_2\operatorname{sen} m\varphi\,. \tag{29}$$

A solução geral é então, usando o princípio da superposição, e escolhendo $c_0 = 0$, $d_0 = 1$,

$$\begin{aligned} u(t,r,\theta,\varphi) = u_0(r,\theta,\varphi) + \sum_{\lambda nm} &\exp(-\kappa\lambda^2t)r^{-1/2}\,\times \\ &\times[c_1J_{n+1/2}(\lambda r) + c_2Y_{n+1/2}(\lambda r)]\,\times \\ &\times[b_1P_n^m(\cos\theta) + b_2Q_m^m(\cos\theta)]\,\times \\ &\times[a_1\cos m\varphi + a_2\operatorname{sen} m\varphi]\,, \end{aligned}$$

em que $u_0(r,\theta,\varphi)$ é uma solução da equação de Laplace.

6 Problemas

1. Considere uma esfera de raio a. Calcule a temperatura $u(t,r,\theta,\varphi)$ sendo $u(t,a,\theta,\varphi) = f(\theta,\varphi)$ e $u(0,r,\theta,\varphi) = g(r,\theta,\varphi)$.

(a) A solução finita é,

$$\begin{aligned} u(t,r,\theta,\varphi) = & u_0(r,\theta,\varphi) + \sum_{\lambda nm} \exp(-\kappa\lambda^2 t) r^{-1/2} \times \\ & \times J_{n+1/2}(\lambda r) P_n^m(\cos\theta) \times \\ & \times [a_1 \cos m\varphi + a_2 sen\, m\varphi] \,. \end{aligned}$$

A condição de contorno em $r = a$ nos dá,

$$\begin{aligned} u(t,a,\theta,\varphi) = & f(\theta,\varphi) = u_0(a,\theta,\varphi) \\ & + \sum_{\lambda nm} \exp(-\kappa\lambda^2 t) a^{-1/2} J_{n+1/2}(\lambda a) \times \\ & \times P_n^m(\cos\theta)[a_1 \cos m\varphi + a_2 sen\, m\varphi] \,. \end{aligned}$$

Satisfazemos a condição acima escolhendo,

$$J_{n+1/2}(\lambda_{nj} a) = 0 \,, \quad j = 1, 2, \ldots \tag{30}$$

Essa equação determina os valores possíveis de λ. Também temos,

$$f(\theta,\varphi) = u_0(a,\theta,\varphi) \,. \tag{31}$$

Escrevendo a condição inicial temos,

$$\begin{aligned} u(0,r,\theta,\varphi) = & g(r,\theta,\varphi) = u_0(r,\theta,\varphi) \\ & + \sum_{jnm} r^{-1/2} J_{n+1/2}(\lambda_{nj} r) \times \\ & \times P_n^m(\cos\theta)[a_1 \cos m\varphi + a_2 sen\, m\varphi] \,. \end{aligned}$$

ou,

$$\begin{aligned} & g(r,\theta,\varphi) - u_0(r,\theta,\varphi) = \\ & = \sum_{jnm} r^{-1/2} J_{n+1/2}(\lambda_{nj} r) P_n^m(\cos\theta)[a_1 \cos m\varphi + a_2 sen\, m\varphi] \,. \end{aligned}$$

Temos uma série de Fourier, logo,

$$\sum_{jn} r^{-1/2} J_{n+1/2}(\lambda_{nj} r) P_n(\cos\theta) a_{10} =$$
$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} [g(r,\theta,x) - u_0(r,\theta,x)] dx \,,$$
$$\sum_{jn} r^{-1/2} J_{n+1/2}(\lambda_{nj} r) P_n^m(\cos\theta) a_{1m} =$$
$$= \frac{1}{\pi} \int_{-\pi}^{\pi} [g(r,\theta,x) - u_0(r,\theta,x)] \cos mx \, dx \,,$$
$$\sum_{jn} r^{-1/2} J_{n+1/2}(\lambda_{nj} r) P_n^m(\cos\theta) a_{2m} =$$
$$= \frac{1}{\pi} \int_{-\pi}^{\pi} [g(r,\theta,x) - u_0(r,\theta,x)] sen\, mx \, dx \,, \quad m = 0, 1, 2, \ldots$$

Temos agora séries de polinômios de Legendre e funções associadas de Legendre do primeiro tipo, assim,

$$\sum_{j} r^{-1/2} J_{n+1/2}(\lambda_{nj} r) a_{10} = \frac{2n+1}{2} \int_0^{\pi} P_n(\cos\theta) sen\,\theta \, d\theta \times$$
$$\times \frac{1}{2\pi} \int_{-\pi}^{\pi} [g(r,\theta,x) - u_0(r,\theta,x)] dx \,,$$
$$\sum_{j} r^{-1/2} J_{n+1/2}(\lambda_{nj} r) a_{1m} = \frac{(2n+1)(n-m)!}{2(n+m)!} \int_0^{\pi} P_n^m(\cos\theta) sen\,\theta \, d\theta \times$$
$$\times \frac{1}{\pi} \int_{-\pi}^{\pi} [g(r,\theta,x) - u_0(r,\theta,x)] \cos mx \, dx \,,$$
$$\sum_{j} r^{-1/2} J_{n+1/2}(\lambda_{nj} r) a_{2m} = \frac{(2n+1)(n-m)!}{2(n+m)!} \int_0^{\pi} P_n^m(\cos\theta) sen\,\theta \, d\theta \times$$
$$\times \frac{1}{\pi} \int_{-\pi}^{\pi} [g(r,\theta,x) - u_0(r,\theta,x)] sen\, mx \, dx \,.$$

As expressões acima são agora séries de funções de Bessel. Multiplicando por $r^{3/2} J_{n+1/2}(\lambda_{nk} r)$ *e integrando em* r*,*

$$\sum_j \int_0^a dr\, r J_{n+1/2}(\lambda_{nj} r) J_{n+1/2}(\lambda_{nk} r) a_{10} = \int_0^a dr\, r^{3/2} J_{n+1/2}(\lambda_{nk} r) \times$$
$$\times \frac{2n+1}{2} \int_0^\pi P_n(\cos\theta) sen\,\theta\, d\theta \times$$
$$\times \frac{1}{2\pi} \int_{-\pi}^{\pi} [g(r,\theta,x) - u_0(r,\theta,x)] dx\,,$$
$$\sum_j \int_0^a dr\, r J_{n+1/2}(\lambda_{nj} r) J_{n+1/2}(\lambda_{nk} r) a_{1m} = \int_0^a dr\, r^{3/2} J_{n+1/2}(\lambda_{nk} r) \times$$
$$\times \frac{(2n+1)(n-m)!}{2(n+m)!} \int_0^\pi P_n^m(\cos\theta) sen\,\theta\, d\theta \times$$
$$\times \frac{1}{\pi} \int_{-\pi}^{\pi} [g(r,\theta,x) - u_0(r,\theta,x)] \cos mx\, dx\,,$$
$$\sum_j \int_0^a dr\, r J_{n+1/2}(\lambda_{nj} r) J_{n+1/2}(\lambda_{nk} r) a_{2m} = \int_0^a dr\, r^{3/2} J_{n+1/2}(\lambda_{nk} r) \times$$
$$\times \frac{(2n+1)(n-m)!}{2(n+m)!} \int_0^\pi P_n^m(\cos\theta) sen\,\theta\, d\theta \times$$
$$\times \frac{1}{\pi} \int_{-\pi}^{\pi} [g(r,\theta,x) - u_0(r,\theta,x)] sen\, mx\, dx\,.$$

Usando a condição de ortogonalidade,

$$\int_0^a r J_{n+1/2}(\lambda_{nj} r) J_{n+1/2}(\lambda_{nk} r) dr = \begin{cases} 0\,, & j \neq k\,, \\ \frac{a^2}{2}[J'_{n+1/2}(\lambda_{nj} a)]^2\,, & j = k\,, \end{cases}$$

temos,

$$
\begin{aligned}
&\frac{a^2}{2}[J'_{n+1/2}(\lambda_{nk}a)]^2 a_{10} = \int_0^a dr\, r^{3/2} J_{n+1/2}(\lambda_{nk}r) \times \\
&\times \frac{2n+1}{2} \int_0^\pi P_n(\cos\theta) sen\,\theta\, d\theta \times \\
&\times \frac{1}{2\pi} \int_{-\pi}^{\pi} [g(r,\theta,x) - u_0(r,\theta,x)] dx\,, \\
&\frac{a^2}{2}[J'_{n+1/2}(\lambda_{nk}a)]^2 a_{1m} = \int_0^a dr\, r^{3/2} J_{n+1/2}(\lambda_{nk}r) \times \\
&\times \frac{(2n+1)(n-m)!}{2(n+m)!} \int_0^\pi P_n^m(\cos\theta) sen\,\theta\, d\theta \times \\
&\times \frac{1}{\pi} \int_{-\pi}^{\pi} [g(r,\theta,x) - u_0(r,\theta,x)] \cos mx\, dx\,, \\
&\frac{a^2}{2}[J'_{n+1/2}(\lambda_{nk}a)]^2 a_{2m} = \int_0^a dr\, r^{3/2} J_{n+1/2}(\lambda_{nk}r) \times \\
&\times \frac{(2n+1)(n-m)!}{2(n+m)!} \int_0^\pi P_n^m(\cos\theta) sen\,\theta\, d\theta \times \\
&\times \frac{1}{\pi} \int_{-\pi}^{\pi} [g(r,\theta,x) - u_0(r,\theta,x)] sen\, mx\, dx\,.
\end{aligned}
$$

Os coeficientes na solução são então,

$$
\begin{aligned}
&a_{10} = \frac{2}{a^2[J'_{n+1/2}(\lambda_{nk}a)]^2} \int_0^a dr\, r^{3/2} J_{n+1/2}(\lambda_{nk}r) \times \\
&\times \frac{2n+1}{2} \int_0^\pi P_n(\cos\theta) sen\,\theta\, d\theta \times \\
&\times \frac{1}{2\pi} \int_{-\pi}^{\pi} [g(r,\theta,x) - u_0(r,\theta,x)] dx\,, \\
&a_{1m} = \frac{2}{a^2[J'_{n+1/2}(\lambda_{nk}a)]^2} \int_0^a dr\, r^{3/2} J_{n+1/2}(\lambda_{nk}r) \times \\
&\times \frac{(2n+1)(n-m)!}{2(n+m)!} \int_0^\pi P_n^m(\cos\theta) sen\,\theta\, d\theta \times \\
&\times \frac{1}{\pi} \int_{-\pi}^{\pi} [g(r,\theta,x) - u_0(r,\theta,x)] \cos mx\, dx\,, \\
&a_{2m} = \frac{2}{a^2[J'_{n+1/2}(\lambda_{nk}a)]^2} \int_0^a dr\, r^{3/2} J_{n+1/2}(\lambda_{nk}r) \times \\
&\times \frac{(2n+1)(n-m)!}{2(n+m)!} \int_0^\pi P_n^m(\cos\theta) sen\,\theta\, d\theta \times \\
&\times \frac{1}{\pi} \int_{-\pi}^{\pi} [g(r,\theta,x) - u_0(r,\theta,x)] sen\, mx\, dx\,.
\end{aligned}
$$

A solução é portanto,

$$
\begin{aligned}
u(t,r,\theta,\varphi) = u_0(r,\theta,\varphi) \\
&+ \sum_{jn} \exp(-\kappa\lambda_{nj}^2 t) r^{-1/2} J_{n+1/2}(\lambda_{nj} r) \times \\
&\times P_n(\cos\theta) \times \\
&\times \frac{2}{a^2[J'_{n+1/2}(\lambda_{nj}a)]^2} \int_0^a dr\, r^{3/2} J_{n+1/2}(\lambda_{nj} r) \times \\
&\times \frac{2n+1}{2} \int_0^\pi P_n(\cos\theta) sen\,\theta\, d\theta \times \\
&\times \frac{1}{2\pi} \int_{-\pi}^{\pi} [g(r,\theta,x) - u_0(r,\theta,x)] dx \\
&+ \sum_{jnm} \exp(-\kappa\lambda_{nj}^2 t) r^{-1/2} J_{n+1/2}(\lambda_{nj} r) \times \\
&\times P_n^m(\cos\theta) \cos m\varphi \times \\
&\times \frac{2}{a^2[J'_{n+1/2}(\lambda_{nj}a)]^2} \int_0^a dr\, r^{3/2} J_{n+1/2}(\lambda_{nj} r) \times \\
&\times \frac{(2n+1)(n-m)!}{2(n+m)!} \int_0^\pi P_n^m(\cos\theta) sen\,\theta\, d\theta \times \\
&\times \frac{1}{\pi} \int_{-\pi}^{\pi} [g(r,\theta,x) - u_0(r,\theta,x)] \cos mx\, dx \\
&+ \sum_{jnm} \exp(-\kappa\lambda_{nj}^2 t) r^{-1/2} J_{n+1/2}(\lambda_{nj} r) \times \\
&\times P_n^m(\cos\theta) sen\, m\varphi \times \\
&\times \frac{2}{a^2[J'_{n+1/2}(\lambda_{nj}a)]^2} \int_0^a dr\, r^{3/2} J_{n+1/2}(\lambda_{nj} r) \times \\
&\times \frac{(2n+1)(n-m)!}{2(n+m)!} \int_0^\pi P_n^m(\cos\theta) sen\,\theta\, d\theta \times \\
&\times \frac{1}{\pi} \int_{-\pi}^{\pi} [g(r,\theta,x) - u_0(r,\theta,x)] sen\, mx\, dx\,.
\end{aligned}
$$

(b) Se $f(\theta,\varphi) = f(\theta)$ e $g(r,\theta,\varphi) = g(r,\theta)$, temos $u_0(a,\theta,\varphi) = u_0(a,\theta) = f(\theta)$ e,

$$
\begin{aligned}
u(t,r,\theta) = u_0(r,\theta) \\
&+ \sum_{jn} \exp(-\kappa\lambda_{nj}^2 t) r^{-1/2} J_{n+1/2}(\lambda_{nj} r) \times \\
&\times P_n(\cos\theta) \times \\
&\times \frac{2}{a^2[J'_{n+1/2}(\lambda_{nj}a)]^2} \int_0^a dr\, r^{3/2} J_{n+1/2}(\lambda_{nj} r) \times \\
&\times \frac{2n+1}{2} \int_0^\pi P_n(\cos\theta) sen\,\theta\, d\theta \times \\
&\times [g(r,\theta) - u_0(r,\theta)]\,.
\end{aligned}
$$

(c) Se $f = f_0$ constante, $u_0(r,\theta) = f_0$, e se temos também $g = g(r)$,

$$
\begin{aligned}
u(t,r) = {}& f_0 + \sum_j \exp(-\kappa\lambda_{0j}^2 t) r^{-1/2} J_{1/2}(\lambda_{0j} r) \times \\
& \times \frac{2}{a^2 [J'_{1/2}(\lambda_{0j} a)]^2} \int_0^a dr\, r^{3/2} J_{1/2}(\lambda_{0j} r) \times \\
& \times [g(r) - f_0] \,.
\end{aligned}
$$

2. Considere a região $a \leq r \leq b$. Calcule a temperatura $u(t,r,\theta,\varphi)$ sendo $u(t,a,\theta,\varphi) = f(\theta,\varphi)$, $u(t,b,\theta,\varphi) = g(\theta,\varphi)$, $u(0,r,\theta,\varphi) = h(r,\theta,\varphi)$.

(a) Escrevemos agora a solução como,

$$
\begin{aligned}
u(t,r,\theta,\varphi) = {}& u_0(r,\theta,\varphi) + \sum_{\lambda nm} \exp(-\kappa\lambda^2 t) r^{-1/2} \times \\
& \times [c_1 J_{n+1/2}(\lambda r) + c_2 Y_{n+1/2}(\lambda r)] \times \\
& \times P_n^m(\cos\theta)[a_1 \cos m\varphi + a_2 sen\, m\varphi] \,.
\end{aligned}
$$

As condições de contorno em $r = a$ e $r = b$,

$$
\begin{aligned}
u(t,a,\theta,\varphi) = {}& f(\theta,\varphi) = u_0(a,\theta,\varphi) + \sum_{\lambda nm} \exp(-\kappa\lambda^2 t) a^{-1/2} \times \\
& \times [c_1 J_{n+1/2}(\lambda a) + c_2 Y_{n+1/2}(\lambda a)] \times \\
& \times P_n^m(\cos\theta)[a_1 \cos m\varphi + a_2 sen\, m\varphi] \,, \\
u(t,b,\theta,\varphi) = {}& g(\theta,\varphi) = u_0(b,\theta,\varphi) + \sum_{\lambda nm} \exp(-\kappa\lambda^2 t) b^{-1/2} \times \\
& \times [c_1 J_{n+1/2}(\lambda b) + c_2 Y_{n+1/2}(\lambda b)] \times \\
& \times P_n^m(\cos\theta)[a_1 \cos m\varphi + a_2 sen\, m\varphi] \,.
\end{aligned}
$$

Satisfazemos as equações acima escolhendo,

$$
\begin{aligned}
& c_1 J_{n+1/2}(\lambda a) + c_2 Y_{n+1/2}(\lambda a) = 0 \,, \\
& c_1 J_{n+1/2}(\lambda b) + c_2 Y_{n+1/2}(\lambda b) = 0 \,.
\end{aligned}
$$

Portanto, os valores possíveis de λ são definidos por,

$$
J_{n+1/2}(\lambda_{nj} a) Y_{n+1/2}(\lambda_{nj} b) - Y_{n+1/2}(\lambda_{nj} a) J_{n+1/2}(\lambda_{nj} b) = 0 \,, \quad j = 1, 2, \ldots
$$

ou,

$$u_{n+1/2}(\lambda_{nj}b) = 0\,,$$

com,

$$u_{n+1/2}(\lambda_{nj}r) \equiv Y_{n+1/2}(\lambda_{nj}a)J_{n+1/2}(\lambda_{nj}r) - J_{n+1/2}(\lambda_{nj}a)Y_{n+1/2}(\lambda_{nj}r)\,.$$

Também temos,

$$\begin{aligned} f(\theta,\varphi) &= u_0(a,\theta,\varphi)\,,\\ g(\theta,\varphi) &= u_0(b,\theta,\varphi)\,, \end{aligned}$$

e,

$$c_2 = -c_1\frac{J_{n+1/2}(\lambda_{nj}a)}{Y_{n+1/2}(\lambda_{nj}a)}\,.$$

A solução fica então, fazendo $c_1 = 1$*,*

$$\begin{aligned} u(t,r,\theta,\varphi) = u_0(r,\theta,\varphi) &+ \sum_{jnm} \exp(-\kappa\lambda_{nj}^2 t) r^{-1/2} \times\\ &\times u_{n+1/2}(\lambda_{nj}r)\frac{1}{Y_{n+1/2}(\lambda_{nj}a)} \times\\ &\times P_n^m(\cos\theta)[a_1\cos m\varphi + a_2 sen\, m\varphi]\,. \end{aligned}$$

A condição inicial nos dá,

$$\begin{aligned} u(0,r,\theta,\varphi) = h(r,\theta,\varphi) &= u_0(r,\theta,\varphi)\\ &+ \sum_{jnm} r^{-1/2} u_{n+1/2}(\lambda_{nj}r)\frac{1}{Y_{n+1/2}(\lambda_{nj}a)} \times\\ &\times P_n^m(\cos\theta)[a_1\cos m\varphi + a_2 sen\, m\varphi]\,, \end{aligned}$$

ou,

$$\begin{aligned} h(r,\theta,\varphi) - u_0(r,\theta,\varphi) = &\sum_{jnm} r^{-1/2} u_{n+1/2}(\lambda_{nj}r)\frac{1}{Y_{n+1/2}(\lambda_{nj}a)} \times\\ &\times P_n^m(\cos\theta)[a_1\cos m\varphi + a_2 sen\, m\varphi]\,. \end{aligned}$$

A expressão acima é uma série de Fourier, logo,

$$\sum_{jn} r^{-1/2} u_{n+1/2}(\lambda_{nj} r) \frac{1}{Y_{n+1/2}(\lambda_{nj} a)} P_n(\cos\theta) a_{10} =$$
$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} [h(r,\theta,x) - u_0(r,\theta,x)]\, dx\,,$$
$$\sum_{jn} r^{-1/2} u_{n+1/2}(\lambda_{nj} r) \frac{1}{Y_{n+1/2}(\lambda_{nj} a)} P_n^m(\cos\theta) a_{1m} =$$
$$= \frac{1}{\pi} \int_{-\pi}^{\pi} [h(r,\theta,x) - u_0(r,\theta,x)] \cos mx\, dx\,,$$
$$\sum_{jn} r^{-1/2} u_{n+1/2}(\lambda_{nj} r) \frac{1}{Y_{n+1/2}(\lambda_{nj} a)} P_n^m(\cos\theta) a_{2m} =$$
$$= \frac{1}{\pi} \int_{-\pi}^{\pi} [h(r,\theta,x) - u_0(r,\theta,x)] sen\, mx\, dx\,.$$

Temos agora séries de polinômios de Legendre e de funções associadas de Legendre, assim,

$$\sum_{j} r^{-1/2} u_{n+1/2}(\lambda_{nj} r) \frac{1}{Y_{n+1/2}(\lambda_{nj} a)} a_{10} =$$
$$= \frac{2n+1}{2} \int_{0}^{\pi} P_n(\cos\theta) sen\,\theta\, d\theta \times$$
$$\times \frac{1}{2\pi} \int_{-\pi}^{\pi} [h(r,\theta,x) - u_0(r,\theta,x)]\, dx\,,$$
$$\sum_{j} r^{-1/2} u_{n+1/2}(\lambda_{nj} r) \frac{1}{Y_{n+1/2}(\lambda_{nj} a)} a_{1m} =$$
$$= \frac{(2n+1)(n-m)!}{2(n+m)!} \int_{0}^{\pi} P_n^m(\cos\theta) sen\,\theta\, d\theta \times$$
$$\times \frac{1}{\pi} \int_{-\pi}^{\pi} [h(r,\theta,x) - u_0(r,\theta,x)] \cos mx\, dx\,,$$
$$\sum_{j} r^{-1/2} u_{n+1/2}(\lambda_{nj} r) \frac{1}{Y_{n+1/2}(\lambda_{nj} a)} a_{2m} =$$
$$= \frac{(2n+1)(n-m)!}{2(n+m)!} \int_{0}^{\pi} P_n^m(\cos\theta) sen\,\theta\, d\theta \times$$
$$\times \frac{1}{\pi} \int_{-\pi}^{\pi} [h(r,\theta,x) - u_0(r,\theta,x)] sen\, mx\, dx\,.$$

As expressões acima são séries de funções de Bessel. Multiplicando por $r^{3/2} u_{n+1/2}(\lambda_{nk} r)$ *e integrando em* r*,*

$$\sum_j \int_a^b dr\, r u_{n+1/2}(\lambda_{nj} r) u_{n+1/2}(\lambda_{nk} r) \frac{1}{Y_{n+1/2}(\lambda_{nj} a)} a_{10} =$$
$$= \int_a^b dr\, r^{3/2} u_{n+1/2}(\lambda_{nk} r) \frac{2n+1}{2} \int_0^\pi P_n(\cos\theta) sen\,\theta\, d\theta \times$$
$$\times \frac{1}{2\pi} \int_{-\pi}^{\pi} [h(r,\theta,x) - u_0(r,\theta,x)]\, dx\,,$$
$$\sum_j \int_a^b dr\, r u_{n+1/2}(\lambda_{nj} r) u_{n+1/2}(\lambda_{nk} r) \frac{1}{Y_{n+1/2}(\lambda_{nj} a)} a_{1m} =$$
$$= \int_a^b dr\, r^{3/2} u_{n+1/2}(\lambda_{nk} r) \frac{(2n+1)(n-m)!}{2(n+m)!} \int_0^\pi P_n^m(\cos\theta) sen\,\theta\, d\theta \times$$
$$\times \frac{1}{\pi} \int_{-\pi}^{\pi} [h(r,\theta,x) - u_0(r,\theta,x)] \cos mx\, dx\,,$$
$$\sum_j \int_a^b dr\, r u_{n+1/2}(\lambda_{nj} r) u_{n+1/2}(\lambda_{nk} r) \frac{1}{Y_{n+1/2}(\lambda_{nj} a)} a_{2m} =$$
$$= \int_a^b dr\, r^{3/2} u_{n+1/2}(\lambda_{nk} r) \frac{(2n+1)(n-m)!}{2(n+m)!} \int_0^\pi P_n^m(\cos\theta) sen\,\theta\, d\theta \times$$
$$\times \frac{1}{\pi} \int_{-\pi}^{\pi} [h(r,\theta,x) - u_0(r,\theta,x)] sen\, mx\, dx\,.$$

Usando a relação de ortogonalidade,

$$\int_a^b r u_{n+1/2}(\lambda_{nj} r) u_{n+1/2}(\lambda_{nk} r) dr = 0\,,\quad j \neq k\,,$$

temos,

$$
\begin{aligned}
&\int_a^b dr\, r u^2_{n+1/2}(\lambda_{nk} r)\frac{1}{Y_{n+1/2}(\lambda_{nk}a)} a_{10} = \\
&= \int_a^b dr\, r^{3/2} u_{n+1/2}(\lambda_{nk} r)\frac{2n+1}{2}\int_0^\pi P_n(\cos\theta) sen\,\theta\, d\theta \times \\
&\times \frac{1}{2\pi}\int_{-\pi}^{\pi}[h(r,\theta,x) - u_0(r,\theta,x)]\, dx\,, \\
&\int_a^b dr\, r u^2_{n+1/2}(\lambda_{nk} r)\frac{1}{Y_{n+1/2}(\lambda_{nk}a)} a_{1m} = \\
&= \int_a^b dr\, r^{3/2} u_{n+1/2}(\lambda_{nk} r)\frac{(2n+1)(n-m)!}{2(n+m)!}\int_0^\pi P_n^m(\cos\theta) sen\,\theta\, d\theta \times \\
&\times \frac{1}{\pi}\int_{-\pi}^{\pi}[h(r,\theta,x) - u_0(r,\theta,x)] \cos mx\, dx\,, \\
&\int_a^b dr\, r u^2_{n+1/2}(\lambda_{nk} r)\frac{1}{Y_{n+1/2}(\lambda_{nk}a)} a_{2m} = \\
&= \int_a^b dr\, r^{3/2} u_{n+1/2}(\lambda_{nk} r)\frac{(2n+1)(n-m)!}{2(n+m)!}\int_0^\pi P_n^m(\cos\theta) sen\,\theta\, d\theta \times \\
&\times \frac{1}{\pi}\int_{-\pi}^{\pi}[h(r,\theta,x) - u_0(r,\theta,x)] sen\, mx\, dx\,.
\end{aligned}
$$

Os coeficientes na solução são então,

$$
\begin{aligned}
a_{10} &= \frac{Y_{n+1/2}(\lambda_{nk}a)}{\int_a^b dr\, r u^2_{n+1/2}(\lambda_{nk} r)} \times \\
&\times \int_a^b dr\, r^{3/2} u_{n+1/2}(\lambda_{nk} r)\frac{2n+1}{2}\int_0^\pi P_n(\cos\theta) sen\,\theta\, d\theta \times \\
&\times \frac{1}{2\pi}\int_{-\pi}^{\pi}[h(r,\theta,x) - u_0(r,\theta,x)]\, dx\,, \\
a_{1m} &= \frac{Y_{n+1/2}(\lambda_{nk}a)}{\int_a^b dr\, r u^2_{n+1/2}(\lambda_{nk} r)} \times \\
&\times \int_a^b dr\, r^{3/2} u_{n+1/2}(\lambda_{nk} r)\frac{(2n+1)(n-m)!}{2(n+m)!}\int_0^\pi P_n^m(\cos\theta) sen\,\theta\, d\theta \times \\
&\times \frac{1}{\pi}\int_{-\pi}^{\pi}[h(r,\theta,x) - u_0(r,\theta,x)] \cos mx\, dx\,, \\
a_{2m} &= \frac{Y_{n+1/2}(\lambda_{nk}a)}{\int_a^b dr\, r u^2_{n+1/2}(\lambda_{nk} r)} \times \\
&\times \int_a^b dr\, r^{3/2} u_{n+1/2}(\lambda_{nk} r)\frac{(2n+1)(n-m)!}{2(n+m)!}\int_0^\pi P_n^m(\cos\theta) sen\,\theta\, d\theta \times \\
&\times \frac{1}{\pi}\int_{-\pi}^{\pi}[h(r,\theta,x) - u_0(r,\theta,x)] sen\, mx\, dx\,.
\end{aligned}
$$

Com isso a solução fica,

$$
\begin{aligned}
u(t,r,\theta,\varphi) = u_0(r,\theta,\varphi) + \sum_{jn} \exp(-\kappa\lambda_{nj}^2 t) r^{-1/2} u_{n+1/2}(\lambda_{nj} r) \times \\
\times P_n(\cos\theta) \frac{1}{\int_a^b dr\, r u_{n+1/2}^2(\lambda_{nj} r)} \times \\
\times \int_a^b dr\, r^{3/2} u_{n+1/2}(\lambda_{nj} r) \times \\
\times \frac{2n+1}{2} \int_0^\pi P_n(\cos\theta) sen\,\theta\, d\theta \times \\
\times \frac{1}{2\pi} \int_{-\pi}^{\pi} [h(r,\theta,x) - u_0(r,\theta,x)]\, dx \\
+ \sum_{jnm} \exp(-\kappa\lambda_{nj}^2 t) r^{-1/2} u_{n+1/2}(\lambda_{nj} r) \times \\
\times P_n^m(\cos\theta) \cos m\varphi \frac{1}{\int_a^b dr\, r u_{n+1/2}^2(\lambda_{nj} r)} \times \\
\times \int_a^b dr\, r^{3/2} u_{n+1/2}(\lambda_{nj} r) \times \\
\times \frac{(2n+1)(n-m)!}{2(n+m)!} \int_0^\pi P_n^m(\cos\theta) sen\,\theta\, d\theta \times \\
\times \frac{1}{\pi} \int_{-\pi}^{\pi} [h(r,\theta,x) - u_0(r,\theta,x)] \cos mx\, dx \\
+ \sum_{jnm} \exp(-\kappa\lambda_{nj}^2 t) r^{-1/2} u_{n+1/2}(\lambda_{nj} r) \times \\
\times P_n^m(\cos\theta) sen\, m\varphi \frac{1}{\int_a^b dr\, r u_{n+1/2}^2(\lambda_{nj} r)} \times \\
\times \int_a^b dr\, r^{3/2} u_{n+1/2}(\lambda_{nj} r) \times \\
\times \frac{(2n+1)(n-m)!}{2(n+m)!} \int_0^\pi P_n^m(\cos\theta) sen\,\theta\, d\theta \times \\
\times \frac{1}{\pi} \int_{-\pi}^{\pi} [h(r,\theta,x) - u_0(r,\theta,x)] sen\, mx\, dx\,.
\end{aligned}
$$

(b) Se temos $f(\theta,\varphi) = f(\theta)$, $g(\theta,\varphi) = g(\theta)$, $h(r,\theta,\varphi) = h(r,\theta)$, $u_0(r,\theta,\varphi) = u_0(r,\theta)$,

$$
\begin{aligned}
u(t,r,\theta) = u_0(r,\theta) + \sum_{jn} & \exp(-\kappa\lambda_{nj}^2 t) r^{-1/2} u_{n+1/2}(\lambda_{nj} r) \times \\
& \times P_n(\cos\theta) \frac{1}{\displaystyle\int_a^b dr\, r u_{n+1/2}^2(\lambda_{nj} r)} \times \\
& \times \int_a^b dr\, r^{3/2} u_{n+1/2}(\lambda_{nj} r) \times \\
& \times \frac{2n+1}{2} \int_0^\pi P_n(\cos\theta) sen\,\theta\, d\theta [h(r,\theta) - u_0(r,\theta)] \,.
\end{aligned}
$$

(c) Se temos apenas dependência em r, isto é, $f = f_0$ e $g = g_0$ constantes, e $h = h(r)$, temos,

$$
\begin{aligned}
u(t,r) = u_0(r) & \\
& + \sum_j \exp(-\kappa\lambda_{0j}^2 t) r^{-1/2} u_{1/2}(\lambda_{0j} r) \times \\
& \times \frac{1}{\displaystyle\int_a^b dr\, r u_{1/2}^2(\lambda_{0j} r)} \times \\
& \times \int_a^b dr\, r^{3/2} u_{1/2}(\lambda_{0j} r) [h(r) - u_0(r)] \,.
\end{aligned}
$$

3. Calcule $u(t,r,\theta,\varphi)$ em $0 \leq r \leq a$ com as condições,

$$
\begin{aligned}
u(t,a,\theta,\varphi) &= \mu(t,\theta,\varphi) \,, \\
u(0,a,\theta,\varphi) &= \varphi(\theta,\varphi) \,.
\end{aligned}
$$

4. Calcule $u(t,r,\theta,\varphi)$ em $a \leq r \leq b$ com as condições,

$$
\begin{aligned}
u(t,a,\theta,\varphi) &= \mu_1(t,\theta,\varphi) \,, \\
u(t,b,\theta,\varphi) &= \mu_2(t,\theta,\varphi) \,, \\
u(0,a,\theta,\varphi) &= \varphi(\theta,\varphi) \,.
\end{aligned}
$$

7 Apêndice

(a) Série de Fourier

$$
\begin{aligned}
f(x) &= \frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \mathrm{sen}\, mx)\,, \\
\frac{a_0}{2} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx\,, \\
a_m &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx\, dx\,, \\
b_m &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \mathrm{sen}\, mx\, dx\,, \quad m = 0, 1, 2, \ldots
\end{aligned}
$$

(b) Polinômios de Legendre ($x = \cos\theta$)

$$
\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= \frac{1}{2}(3x^2 - 1) \\
P_3(x) &= \frac{1}{2}(5x^3 - 3x) \\
P_4(x) &= \frac{1}{8}(35x^4 - 30x^2 + 3) \\
P_5(x) &= \frac{1}{8}(63x^5 - 70x^3 + 15x) \\
P_6(x) &= \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5) \\
P_7(x) &= \frac{1}{16}(497x^7 - 693x^5 + 315x^3 - 35x) \\
P_8(x) &= \frac{1}{128}(6435x^8 - 12012x^6 + 6930x^4 - 1260x^2 + 35)
\end{aligned}
$$

Relação de ortogonalidade:

$$
\int_{-1}^{+1} P_n(x) P_k(x) dx = \begin{cases} 0\,, & n \neq k\,, \\ \dfrac{2}{2n+1}\,, & n = k\,, \end{cases}
$$

(c) Funções associadas de Legendre

$$
\begin{aligned}
&P_1^1(x) = (1-x^2)^{1/2} = \operatorname{sen}\theta \\
&P_2^1(x) = 3x(1-x^2)^{1/2} = 3\operatorname{sen}\theta\cos\theta \\
&P_2^2(x) = 3(1-x^2) = 3\operatorname{sen}^2\theta \\
&P_3^1(x) = \frac{3}{2}(5x^2-1)(1-x^2)^{1/2} = \frac{3}{2}(5\cos^2\theta-1)\operatorname{sen}\theta \\
&P_3^2(x) = 15x(1-x^2) = 15\operatorname{sen}^2\theta\cos\theta \\
&P_3^3(x) = 15(1-x^2)^{3/2} = 15\operatorname{sen}^3\theta \\
&P_4^1(x) = \frac{5}{2}(7x^3-3x)(1-x^2)^{1/2} = \frac{5}{2}(7\cos^3\theta-3\cos\theta)\operatorname{sen}\theta \\
&P_4^2(x) = \frac{15}{2}(7x^2-1)(1-x^2) = \frac{15}{2}(7\cos^2\theta-1)\operatorname{sen}^2\theta \\
&P_4^3(x) = 105x(1-x^2)^{3/2} = 105\operatorname{sen}^3\theta\cos\theta \\
&P_4^4(x) = 105(1-x^2)^2 = 105\operatorname{sen}^4\theta
\end{aligned}
$$

Relação de ortogonalidade:

$$
\int_{-1}^{+1} P_n^m(x)P_k^m(x)dx = \begin{cases} 0\,, & n \neq k\,, \\ \dfrac{2}{2n+1}\dfrac{(n+m)!}{(n-m)!}\,, & n = k\,, \end{cases}
$$

(d) Série de polinômios de Legendre

$$
\begin{aligned}
&f(\theta) = \sum_{k=0}^{\infty} C_k P_k(\cos\theta)\,, \\
&C_k = \frac{2k+1}{2}\int_0^{\pi} f(\theta)P_k(\cos\theta)\operatorname{sen}\theta\, d\theta\,.
\end{aligned}
$$

(e) Série de funções associadas de Legendre do primeiro tipo

$$
\begin{aligned}
&f(\theta) = \sum_{k=0}^{\infty} D_k P_k^m(\cos\theta)\,, \\
&D_k = \frac{(2k+1)(k-m)!}{2(k+m)!}\int_0^{\pi} f(\theta)P_k^m(\cos\theta)\operatorname{sen}\theta\, d\theta\,,
\end{aligned}
$$

(f) Funções de Bessel

$$
\int_0^a rJ_{1/2}(\lambda_j r)J_{1/2}(\lambda_k r)dr = \begin{cases} 0\,, & j \neq k\,, \\ \dfrac{a^2}{2}[J'_{1/2}(\lambda_k a)]^2\,, & j = k\,, \end{cases}
$$

$$\int_0^a rJ_{n+1/2}(\lambda_{nj}r)J_{n+1/2}(\lambda_{nk}r)dr = \begin{cases} 0\,, & j \neq k\,, \\ \dfrac{a^2}{2}[J'_{n+1/2}(\lambda_{nk}a)]^2\,, & j = k\,, \end{cases}$$

$$u_{1/2}(\lambda_j r) \equiv Y_{1/2}(\lambda_j a)J_{1/2}(\lambda_j r) - J_{1/2}(\lambda_j a)Y_{1/2}(\lambda_j r)\,,$$

$$\int_a^b ru_{1/2}(\lambda_j r)u_{1/2}(\lambda_k r)dr = 0\,, \quad j \neq k\,.$$

Expansão em série de funções $u_{1/2}(\lambda_j r)$,

$$f(r) = \sum_j A_j u_{1/2}(\lambda_j r)\,,$$

$$A_j = \frac{\int_a^b rf(r)u_{1/2}(\lambda_j r)dr}{\int_a^b ru_{1/2}^2(\lambda_j r)dr}\,.$$

$$u_{n+1/2}(\lambda_{nj} r) \equiv Y_{n+1/2}(\lambda_{nj} a)J_{n+1/2}(\lambda_{nj} r) - J_{n+1/2}(\lambda_{nj} a)Y_{n+1/2}(\lambda_{nj} r)\,,$$

$$\int_a^b ru_{n+1/2}(\lambda_{nj} r)u_{n+1/2}(\lambda_{nk} r)dr = 0\,, \quad j \neq k\,.$$

Expansão em série de funções $u_{n+1/2}(\lambda_{nj} r)$,

$$f(r) = \sum_j A_j u_{n+1/2}(\lambda_{nj} r)\,,$$

$$A_j = \frac{\int_a^b rf(r)u_{n+1/2}(\lambda_{nj} r)dr}{\int_a^b ru_{n+1/2}^2(\lambda_{nj} r)dr}\,.$$

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