7 - A equação do calor em coordenadas cilíndricas

A equação do calor é,

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u\,, \tag{1}$$

em que u é a temperatura e κ é a difusividade, que consideraremos aqui ser constante. Em coordenadas cilíndricas,

$$\frac{1}{\kappa}\frac{\partial u}{\partial t} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial u}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2}\,. \tag{2}$$

Primeiro separamos o tempo das coordenadas espaciais, escrevendo a solução como,

$$u(t, \mathbf{r}) = T(t)F(\mathbf{r})\,. \tag{3}$$

Substituindo em (1) obtemos,

$$F\frac{dT}{dt} = \kappa T \nabla^2 F\,,$$

ou,

$$\frac{1}{\kappa T}\frac{dT}{dt} = \frac{\nabla^2 F}{F}\,.$$

Igualando a equação acima a uma constante,

$$\frac{1}{\kappa T}\frac{dT}{dt} = \frac{\nabla^2 F}{F} = -\lambda^2\,,$$

obtemos as equações,

$$\frac{dT}{dt} + \lambda^2 \kappa T = 0\,, \tag{4}$$

e,

$$\nabla^2 F + \lambda^2 F = 0\,. \tag{5}$$

Essa equação é chamada *equação de Helmholtz*. A equação para T possui solução,

$$T(t) = A\exp(-\lambda^2 \kappa t)\,. \tag{6}$$

Temos agora que resolver a equação para F.

1 Considerando $F(\rho)$

A equação para F,

$$\nabla^2 F + \lambda^2 F = 0\,,$$

fica na forma,

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial F}{\partial\rho}\right) + \lambda^2 F = 0\,.$$

Como F depende apenas de ρ,

$$(\rho F')' + \lambda^2\rho F = 0\,,$$

em que $F' = dF/d\rho$. Temos assim a equação,

$$\rho F'' + F' + \lambda^2\rho F = 0\,,$$

ou,

$$\rho^2 F'' + \rho F' + \lambda^2\rho^2 F = 0\,, \tag{7}$$

que é a equação diferencial de Bessel, com solução,

$$F(\rho) = a_1 J_0(\lambda\rho) + a_2 Y_0(\lambda\rho)\,. \tag{8}$$

A solução geral é então,

$$u(t,\rho) = T(t)F(\rho) = \exp(-\lambda^2\kappa t)[a_1 J_0(\lambda\rho) + a_2 Y_0(\lambda\rho)]\,, \tag{9}$$

em que fizemos $A = 1$. O valor das constantes depende das condições de contorno, e usando o princípio da superposição escrevemos,

$$u(t,\rho) = T(t)F(\rho) = \sum_j \exp(-\lambda_j^2\kappa t)[a_{1j} J_0(\lambda_j\rho) + a_{2j} Y_0(\lambda_j\rho)]\,. \tag{10}$$

No caso estacionário temos a equação de Laplace, com solução $u_e(\rho)$. A solução mais geral é então,

$$u(t,\rho) = u_e(\rho) + \sum_j \exp(-\lambda_j^2\kappa t)[a_{1j} J_0(\lambda_j\rho) + a_{2j} Y_0(\lambda_j\rho)]\,. \tag{11}$$

Portanto, se $t \to \infty$, a solução é estacionária, dada pela solução da equação de Laplace. Esse caso corresponde a $\lambda = 0$.

2 Problemas

1. Considere a região $0 \leq \rho \leq a$. Calcule a temperatura $u(t,\rho)$ sendo $u(t,a) = u_a$ e $u(0,\rho) = f(\rho)$.

A solução geral é,

$$u(t,\rho) = c_0 + d_0 \ln \rho + \sum_j A_j \exp(-\lambda_j^2 \kappa t) J_0(\lambda_j \rho)\,,$$

em que $u_e = c_0 + d_0 \ln \rho$ é a solução da equação de Laplace, dependendo apenas de ρ, satisfazendo as condições de contorno apropriadas. Não temos o termo em Y_0 para que a solução seja finita em $\rho = 0$. Pelo mesmo motivo podemos fazer $d_0 = 0$.

As condições de contorno nos dão,

$$u(t,a) = c_0 + \sum_j A_j \exp(-\lambda_j^2 \kappa t) J_0(\lambda_j a) = u_a\,,$$
$$u(0,\rho) = c_0 + \sum_j A_j J_0(\lambda_j \rho) = f(\rho)\,.$$

Satisfazemos as equações acima escolhendo,

$$c_0 = u_a\,,$$
$$J_0(\lambda_j a) = 0\,, \quad j = 1, 2, \ldots\,.$$

A solução estacionária, ou seja a solução da equação de Laplace, é assim $u_L(\rho) = u_a$ constante. A condição de contorno para f fica então,

$$f(\rho) = u_a + \sum_j A_j J_0(\lambda_j \rho)\,,$$

ou,

$$f(\rho) - u_a = \sum_j A_j J_0(\lambda_j \rho)\,.$$

A expressão acima é uma expansão em série de funções de Bessel, logo,

$$A_j = \frac{2}{a^2 J_1^2(\lambda_j a)} \int_0^a \rho J_0(\lambda_j \rho)[f(\rho) - u_a] d\rho\,.$$

A solução é então,

$$
\begin{aligned}
u(t,\rho) &= u_a + \sum_j \exp(-\lambda_j^2\kappa t)J_0(\lambda_j\rho)\times \\
&\quad \times \frac{2}{a^2J_1^2(\lambda_j a)}\int_0^a \rho J_0(\lambda_j\rho)[f(\rho)-u_a]d\rho\,.
\end{aligned}
$$

Consideremos o caso particular $u_a = 0$ (Spiegel [7], probl. 6.28). Nesse caso temos,

$$
u(t,\rho) = \sum_j \exp(-\lambda_j^2\kappa t)J_0(\lambda_j\rho)\frac{2}{a^2J_1^2(\lambda_j a)}\int_0^a \rho J_0(\lambda_j\rho)f(\rho)d\rho\,.
$$

Se temos $f(\rho) = 0$ e $u_a \neq 0$ (Spiegel [7], probl. 6.91), então,

$$
\begin{aligned}
u(t,\rho) &= u_a - u_a\sum_j \exp(-\lambda_j^2\kappa t)J_0(\lambda_j\rho)\times \\
&\quad \times \frac{2}{a^2J_1^2(\lambda_j a)}\int_0^a \rho J_0(\lambda_j\rho)d\rho\,.
\end{aligned}
$$

Como,

$$
\int_0^a \rho J_0(\lambda_j\rho)d\rho = \frac{a}{\lambda_j}J_1(\lambda_j a)\,,
$$

vem,

$$
u(t,\rho) = u_a - u_a\sum_j \exp(-\lambda_j^2\kappa t)\frac{2J_0(\lambda_j\rho)}{\lambda_j a J_1(\lambda_j a)}\,.
$$

Considerando ainda $u_a = 0$, com $f(\rho) = u_0(a^2-\rho^2)$ (Spiegel [7], probl. 6.92),

$$
\begin{aligned}
u(t,\rho) &= \sum_j \exp(-\lambda_j^2\kappa t)J_0(\lambda_j\rho)\frac{2}{a^2J_1^2(\lambda_j a)}\int_0^a \rho J_0(\lambda_j\rho)f(\rho)d\rho\,, \\
&= \sum_j \exp(-\lambda_j^2\kappa t)J_0(\lambda_j\rho)\frac{2u_0}{a^2J_1^2(\lambda_j a)}\int_0^a \rho J_0(\lambda_j\rho)(a^2-\rho^2)d\rho\,, \\
&= 4u_0\sum_j \exp(-\lambda_j^2\kappa t)\frac{J_0(\lambda_j\rho)J_2(\lambda_j a)}{\lambda_j^2J_1^2(\lambda_j a)}\,,
\end{aligned}
$$

em que usamos,

$$\int xJ_0(x)dx = xJ_1(x)\,,$$
$$\int x^3J_0(x)dx = x^3J_1(x) + 2x^2J_0(x) - 4xJ_1(x)\,,$$
$$J_2(x) = \frac{2}{x}J_1(x) - J_0(x)\,.$$

2. Consideremos a região $a \le \rho \le b$. Calcule a temperatura $u(t,\rho)$ sendo $u(t,a) = u_a$, $u(t,b) = u_b$ e $u(0,\rho) = f(\rho)$.

A solução agora inclui $\ln\rho$ *e* $Y_0(x)$*,*

$$u(t,\rho) = c_0 + d_0\ln\rho + \sum_j \exp(-\lambda_j^2\kappa t)[A_jJ_0(\lambda_j\rho) + B_jY_0(\lambda_j\rho)]\,.$$

Escrevendo as condições de contorno,

$$\begin{aligned}
u(t,a) &= u_a = c_0 + d_0\ln a + \sum_j \exp(-\lambda_j^2\kappa t)[A_jJ_0(\lambda_j a) + B_jY_0(\lambda_j a)]\,,\\
u(t,b) &= u_b = c_0 + d_0\ln b + \sum_j \exp(-\lambda_j^2\kappa t)[A_jJ_0(\lambda_j b) + B_jY_0(\lambda_j b)]\,,\\
u(0,\rho) &= f(\rho) = c_0 + d_0\ln\rho + \sum_j [A_jJ_0(\lambda_j\rho) + B_jY_0(\lambda_j\rho)]\,.
\end{aligned}$$

Podemos satisfazer as equações acima escolhendo,

$$A_jJ_0(\lambda_j a) + B_jY_0(\lambda_j a) = 0\,,$$
$$A_jJ_0(\lambda_j b) + B_jY_0(\lambda_j b) = 0\,,$$

logo,

$$u_a = c_0 + d_0\ln a\,,$$
$$u_b = c_0 + d_0\ln b\,.$$

Das equações acima temos,

$$J_0(\lambda_j a)Y_0(\lambda_j b) - J_0(\lambda_j b)Y_0(\lambda_j a) = 0\,.$$

Essa equação determina os valores possíveis de λ_j. Definindo,

$$U_0(\lambda_j\rho) \equiv J_0(\lambda_j\rho)Y_0(\lambda_j a) - J_0(\lambda_j a)Y_0(\lambda_j\rho)\,,$$

temos,

$$U_0(\lambda_j b) = 0\,, \quad j = 1, 2, \ldots$$

Calculando as constantes c_0 e d_0 obtemos,

$$\begin{aligned} c_0 &= \frac{u_b \ln a - u_a \ln b}{\ln a/b}\,, \\ d_0 &= \frac{u_a - u_b}{\ln a/b}\,. \end{aligned}$$

A condição para f fica,

$$f(\rho) - c_0 - d_0 \ln\rho = \sum_j [A_j J_0(\lambda_j\rho) + B_j Y_0(\lambda_j\rho)]\,.$$

Substituindo,

$$B_j = -A_j \frac{J_0(\lambda_j a)}{Y_0(\lambda_j a)}\,,$$

temos,

$$\begin{aligned} f(\rho) - c_0 - d_0 \ln\rho &= \sum_j A_j [J_0(\lambda_j\rho)Y_0(\lambda_j a) - J_0(\lambda_j a)Y_0(\lambda_j\rho)] \frac{1}{Y_0(\lambda_j a)}\,, \\ &= \sum_j \frac{A_j}{Y_0(\lambda_j a)} U_0(\lambda_j\rho)\,. \end{aligned}$$

A equação acima é uma expansão em série em funções de Bessel, logo,

$$\frac{A_j}{Y_0(\lambda_j a)} = \frac{\displaystyle\int_a^b \rho[f(\rho) - c_0 - d_0 \ln\rho] U_0(\lambda_j\rho) d\rho}{\int_a^b \rho[U_0(\lambda_j\rho)]^2 d\rho}\,.$$

A solução do problema é então,

$$
\begin{aligned}
u(t,\rho) &= c_0 + d_0 \ln \rho \\
&+ \sum_j \exp(-\lambda_j^2 \kappa t) U_0(\lambda_j \rho) \frac{\displaystyle\int_a^b \rho[f(\rho) - c_0 - d_0 \ln \rho] U_0(\lambda_j \rho) d\rho}{\displaystyle\int_a^b \rho [U_0(\lambda_j \rho)]^2 d\rho} .
\end{aligned}
$$

A solução estacionária é,

$$u_e(\rho) = c_0 + d_0 \ln \rho .$$

Em $\rho = a$,

$$u_e(a) = c_0 + d_0 \ln a = u_a ,$$

e em $\rho = b$,

$$u_e(b) = c_0 + d_0 \ln b = u_b ,$$

como deve ser.

Consideremos o caso particular $u_a = u_b = 0$ (Spiegel [7] probl. 6.34). Temos então $c_0 = d_0 = 0$ e,

$$u(t,\rho) = \sum_j \exp(-\lambda_j^2 \kappa t) U_0(\lambda_j \rho) \frac{\displaystyle\int_a^b \rho f(\rho) U_0(\lambda_j \rho) d\rho}{\displaystyle\int_a^b \rho [U_0(\lambda_j \rho)]^2 d\rho} .$$

3. Calcule $u(\rho, t)$ para a equação do calor,

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u ,$$

com,

$$
\begin{aligned}
&0 \le \rho \le a , \quad t > 0 , \\
&u_\rho(a,t) = -h[u(a,t) - u_0] , \quad u(\rho, 0) = \varphi(\rho) .
\end{aligned}
$$

4. Calcule $u(\rho, t)$ para a equação do calor,

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u ,$$

com,

$$
\begin{aligned}
&a \leq \rho \leq b\,, \quad t > 0\,,\\
&u_\rho(a,t) = \mu_1(t)\,,\\
&u_\rho(b,t) = \mu_2(t)\,,\\
&u(\rho,0) = \varphi(\rho)\,.
\end{aligned}
$$

5. Calcule $u(\rho,t)$ para a equação do calor,

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u\,,$$

com,

$$
\begin{aligned}
&a \leq \rho \leq b\,, \quad t > 0\,,\\
&u_\rho(a,t) = -h_1[u(a,t) - u_0]\,,\\
&u_\rho(b,t) = -h_2[u(b,t) - u_0]\,,\\
&u(\rho,0) = \varphi(\rho)\,.
\end{aligned}
$$

6. Calcule $u(t,\rho)$ para:

(a)

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + F(\rho)\,,$$

(b)

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + \alpha^2\,,$$

com α constante,

(c)

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + \alpha^2 u\,,$$

com α constante e,

$$
\begin{aligned}
&0 \leq \rho \leq a\,,\\
&u(t,a) = u_a\,,\\
&u(0,\rho) = g(\rho)\,.
\end{aligned}
$$

7. Calcule $u(t,\rho)$ para:

(a)

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + F(t, \rho)\,,$$

(b)

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + \alpha^2\,,$$

com α constante,

(c)

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + \alpha^2 u\,,$$

com α constante e,

$$\begin{aligned} &0 \leq \rho \leq a\,, \\ &u(t, a) = \mu(t)\,, \\ &u(0, \rho) = g(\rho)\,. \end{aligned}$$

8. Calcule $u(t, \rho)$ para:

(a)

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + F(\rho)\,,$$

(b)

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + \alpha^2\,,$$

com α constante,

(c)

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + \alpha^2 u\,,$$

com α constante e,

$$\begin{aligned} &a \leq \rho \leq b\,, \\ &u(t, a) = u_a\,, \quad u(t, b) = u_b\,, \\ &u(0, \rho) = g(\rho)\,. \end{aligned}$$

9. Calcule $u(t, \rho)$ para:

(a)

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + F(t, \rho)\,,$$

(b)

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u\,,$$

(c)

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + \alpha^2\,,$$

com α constante,

(d)

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + \alpha^2 u\,,$$

com α constante e,

$$\begin{aligned}
& a \leq \rho \leq b\,, \\
& u(t, a) = \mu_1(t)\,, \quad u(t, b) = \mu_2(t)\,, \\
& u(0, \rho) = g(\rho)\,.
\end{aligned}$$

3 Considerando $F(\rho, z)$

A função $F(\rho, z)$ satisfaz a equação,

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial F}{\partial \rho}\right) + \frac{\partial^2 F}{\partial z^2} + \lambda^2 F = 0\,. \tag{12}$$

Substituindo $F(\rho, z) = R(\rho) Z(z)$,

$$\frac{Z}{\rho}(\rho R')' + R Z'' + \lambda^2 R Z = 0\,,$$

ou,

$$\frac{1}{\rho R}(\rho R')' + \frac{Z''}{Z} + \lambda^2 = 0\,.$$

Escrevemos a equação acima como,

$$\frac{1}{\rho R}(\rho R')' + \lambda^2 = -\frac{Z''}{Z} \equiv \mu^2\,. \tag{13}$$

A equação para R é então,

$$\rho^2 R'' + \rho R' + (\lambda^2 - \mu^2)\rho^2 R = 0\,,$$

e a equação para Z é,

$$Z'' + \mu^2 Z = 0\,.$$

A função R é,

$$R(\rho) = c_1 J_0(\alpha\rho) + c_2 Y_0(\alpha\rho)\,, \quad \alpha = \sqrt{\lambda^2 - \mu^2}\,, \quad \lambda^2 > \mu^2\,,$$
$$R(\rho) = c_1 I_0(\alpha\rho) + c_2 K_0(\alpha\rho)\,, \quad \alpha = \sqrt{\mu^2 - \lambda^2}\,, \quad \lambda^2 < \mu^2\,,$$

e a função Z é,

$$Z(z) = b_1 \cos\mu z + b_2 \mathrm{sen}\,\mu z\,.$$

A solução $F(\rho, z)$ é assim,

$$\begin{aligned}
F(\rho, z) &= [c_1 J_0(\alpha\rho) + c_2 Y_0(\alpha\rho)][b_1 \cos\mu z + b_2 \mathrm{sen}\,\mu z]\,, \\
&\quad \alpha = \sqrt{\lambda^2 - \mu^2}\,, \quad \lambda^2 > \mu^2\,, \\
F(\rho, z) &= [c_1 I_0(\alpha\rho) + c_2 K_0(\alpha\rho)][b_1 \cos\mu z + b_2 \mathrm{sen}\,\mu z]\,, \\
&\quad \alpha = \sqrt{\mu^2 - \lambda^2}\,, \quad \lambda^2 < \mu^2\,.
\end{aligned}$$

Se $\mu = 0$ temos $Z(z) = a_0 + b_0 z$ e $\alpha = \lambda$, e a solução é,

$$F(\rho, z) = [c_1 J_0(\lambda\rho) + c_2 Y_0(\lambda\rho)](a_0 + b_0 z)\,, \tag{14}$$

Consideremos agora a outra possibilidade para (13),

$$\frac{1}{\rho R}(\rho R')' + \lambda^2 = -\frac{Z''}{Z} \equiv -\mu^2\,. \tag{15}$$

A equação para R é agora,

$$\rho^2 R'' + \rho R' + (\lambda^2 + \mu^2)\rho^2 R = 0\,,$$

com solução,

$$R(\rho) = c_1 J_0(\alpha\rho) + c_2 Y_0(\alpha\rho)\,, \quad \alpha = \sqrt{\lambda^2 + \mu^2}\,.$$

A equação para Z é,

$$Z'' - \mu^2 Z = 0\,,$$

com solução,

$$Z(z) = b_1 \text{cosh}\mu z + b_2 \text{senh}\mu z\,.$$

A solução $F(\rho, z)$ é agora,

$$\begin{aligned} F(\rho, z) &= [c_1 J_0(\alpha\rho) + c_2 Y_0(\alpha\rho)][b_1 \text{cosh}\mu z + b_2 \text{senh}\mu z]\,, \\ & \alpha = \sqrt{\lambda^2 + \mu^2}\,. \end{aligned}$$

Se $\mu = 0$ temos $Z(z) = a_0 + b_0 z$ e $\alpha = \lambda$, e a solução é,

$$F(\rho, z) = [a_1 J_0(\lambda\rho) + a_2 Y_0(\lambda\rho)](a_0 + b_0 z) \tag{16}$$

como na equação (14).

4 Problemas

1. Consideremos a região $0 \leq \rho \leq a$ e $0 \leq z \leq L$. Calcule a temperatura $u(t, \rho, z)$ sendo,

$$\begin{aligned} u(t, a, z) &= f(z)\,, \\ u(t, \rho, 0) &= g(\rho)\,, \\ u(t, \rho, L) &= h(\rho)\,, \\ u(0, \rho, z) &= v(\rho, z)\,. \end{aligned}$$

Vamos escrever a solução das três formas possíveis.
(a) Escrevemos a solução como,

$$u(t, \rho, z) = u_e(\rho, z) + \sum_{\lambda\mu} e^{-\lambda^2 \kappa t} F(\rho, z)\,,$$

com,

$$\begin{aligned} F(\rho, z) &= J_0(\lambda\rho)(a_{0\lambda} + b_{0\lambda} z) \\ & +J_0(\alpha\rho)[b_1 \cos\mu z + b_2 sen\,\mu z]\,, \\ & \alpha = \sqrt{\lambda^2 - \mu^2}\,, \;\; \lambda^2 > \mu^2\,. \end{aligned}$$

e u_e é a solução estacionária. Portanto,

$$
\begin{aligned}
u(t,\rho,z) &= u_e(\rho,z) + \sum_{\lambda\mu} e^{-\lambda^2\kappa t}\left\{J_0(\lambda\rho)(a_{0\lambda}+b_{0\lambda}z)\right.\\
&\left. +J_0(\alpha\rho)[b_1\cos\mu z + b_2 sen\,\mu z]\right\}\,.
\end{aligned}
$$

As condições de contorno são,

$$
\begin{aligned}
u(t,a,z) &= f(z) = u_e(a,z) + \sum_{\lambda\mu} e^{-\lambda^2\kappa t}\left\{J_0(\lambda a)(a_{0\lambda}+b_{0\lambda}z)\right.\\
&\left. +J_0(\alpha a)[b_1\cos\mu z + b_2 sen\,\mu z]\right\}\,,\\
u(t,\rho,0) &= g(\rho) = u_e(\rho,0) + \sum_{\lambda\mu} e^{-\lambda^2\kappa t}\left\{J_0(\lambda\rho)a_{0\lambda}\right.\\
&\left. +J_0(\alpha\rho)b_1\right\}\,,\\
u(t,\rho,L) &= h(\rho) = u_e(\rho,L) + \sum_{\lambda\mu} e^{-\lambda^2\kappa t}\left\{J_0(\lambda\rho)(a_{0\lambda}+b_{0\lambda}L)\right.\\
&\left. +J_0(\alpha\rho)[b_1\cos\mu L + b_2 sen\,\mu L]\right\}\,,\\
u(0,\rho,z) &= v(\rho,z) = u_e(\rho,z) + \sum_{\lambda\mu} \left\{J_0(\lambda\rho)(a_{0\lambda}+b_{0\lambda}z)\right.\\
&\left. +J_0(\alpha\rho)[b_1\cos\mu z + b_2 sen\,\mu z]\right\}\,,\\
&\alpha = \sqrt{\lambda^2-\mu^2}\,,\quad \lambda^2>\mu^2\,.
\end{aligned}
$$

Satisfazemos as condições acima escolhendo,

$$
\begin{aligned}
&a_{0\lambda} = b_{0\lambda} = b_1 = 0\,,\\
&J_0(\alpha_j a) = 0\,,\quad j=1,2,\ldots\\
&\mu_k L = k\pi\ ,,\quad k=1,2,\ldots\\
&\alpha_j^2+\mu_k^2 = \lambda_{jk}^2\,.
\end{aligned}
$$

Com isso temos,

$$
\begin{aligned}
&f(z) = u_e(a,z)\,,\\
&g(\rho) = u_e(\rho,0)\,,\\
&h(\rho) = u_e(\rho,L)\,,\\
&v(\rho,z) = u_e(\rho,z) + \sum_{jk} J_0(\alpha_j\rho) b_2 sen\,\mu_k z\,,\\
&\alpha_j^2+\mu_k^2 = \lambda_{jk}^2\,.
\end{aligned}
$$

A condição para v nos dá uma série de funções de Bessel,

$$v(\rho,z)-u_e(\rho,z)=\sum_{jk}J_0(\alpha_j\rho)b_2 sen\,\mu_k z\,,$$

logo,

$$\sum_k b_2 sen\,\mu_k z=\frac{2}{a^2J_1^2(\alpha_j a)}\int_0^a xJ_0(\alpha_j x)[v(x,z)-u_e(x,z)]dx\,.$$

A expressão acima é uma série de Fourier de senos, assim,

$$b_2=\frac{2}{L}\int_0^L sen\,(k\pi y/L)\,dy\frac{2}{a^2J_1^2(\alpha_j a)}\int_0^a xJ_0(\alpha_j x)[v(x,y)-u_e(x,y)]dx\,.$$

A série para v é portanto,

$$v(\rho,z)-u_e(\rho,z)=\sum_{jk}J_0(\alpha_j\rho)sen\,\mu_k z\times$$
$$\times\frac{2}{L}\int_0^L sen\,(k\pi y/L)\,dy\frac{2}{a^2J_1^2(\alpha_j a)}\int_0^a xJ_0(\alpha_j x)[v(x,y)-u_e(x,y)]dx\,.$$

Se $v-u_e=v_0$ constante,

$$\begin{aligned}
v_0 &= \sum_{jk}J_0(\alpha_j\rho)sen\,\mu_k z\times\\
&\quad\times\frac{2}{L}\int_0^L sen\,(k\pi y/L)\,dy\frac{2}{a^2J_1^2(\alpha_j a)}v_0\int_0^a xJ_0(\alpha_j x)dx\,,\\
&= \sum_{jk}J_0(\alpha_j\rho)sen\,\mu_k z\times\\
&\quad\times\frac{2v_0}{a\alpha_j J_1(\alpha_j a)}\frac{2}{L}\int_0^L sen\,(k\pi y/L)\,dy\,,\\
&= v_0\frac{2}{a}\sum_j\frac{J_0(\alpha_j\rho)}{\alpha_j J_1(\alpha_j a)}\frac{4}{\pi}\sum_k\frac{sen\,[(2k-1)\pi z/L]}{2k-1}=v_0\,,
\end{aligned}$$

como esperado.

A solução é portanto ([7], probl. 6.29 e 6.115 com $u_e=0$),

$$
\begin{aligned}
u(t,\rho,z) &= u_e(\rho,z) + \frac{2}{a^2}\sum_{jk} e^{-(\alpha_j^2+(k\pi/L)^2)\kappa t}\frac{J_0(\alpha_j\rho)}{J_1^2(\alpha_j a)} sen\,(k\pi z/L)\times \\
&\quad \times\frac{2}{L}\int_0^L sen\,(k\pi y/L)\,dy\int_0^a xJ_0(\alpha_j x)[v(x,y)-u_e(x,y)]dx\,.
\end{aligned}
$$

Se $v-u_e=v_0$ constante ([7], probl. 6.30 com $u_e=0$),

$$
\begin{aligned}
u(t,\rho,z) &= u_e(\rho,z) + \frac{2}{a^2}\sum_{jk} e^{-(\alpha_j^2+(k\pi/L)^2)\kappa t}\frac{J_0(\alpha_j\rho)}{J_1^2(\alpha_j a)} sen\,(k\pi z/L)\times \\
&\quad \times\frac{2}{L}\int_0^L sen\,(k\pi y/L)\,dy\, v_0\int_0^a xJ_0(\alpha_j x)dx\,, \\
&= u_e(\rho,z) + \frac{2}{a}\sum_{jk} e^{-(\alpha_j^2+(k\pi/L)^2)\kappa t}\frac{J_0(\alpha_j\rho)}{\alpha_j J_1(\alpha_j a)} sen\,(k\pi z/L)\times \\
&\quad \times\frac{2}{L}\int_0^L sen\,(k\pi y/L)\,dy\, v_0\,, \\
&= u_e(\rho,z) + \frac{8v_0}{a\pi}\sum_{jk} e^{-[\alpha_j^2+((2k-1)\pi/L)^2]\kappa t}\frac{J_0(\alpha_j\rho)}{\alpha_j J_1(\alpha_j a)}\times \\
&\quad \times\frac{sen\,[(2k-1)\pi z/L]}{2k-1}\,.
\end{aligned}
$$

Para $f=0$, $g=g_0$ e $h=h_0$ temos o problema 6.114 em [7].

(b) Escrevemos agora,

$$
u(t,\rho,z) = u_e(\rho,z) + \sum_{\lambda\mu} e^{-\lambda^2\kappa t}F(\rho,z)\,,
$$

com,

$$
\begin{aligned}
F(\rho,z) &= J_0(\lambda\rho)(a_0+b_0 z) \\
&\quad +I_0(\alpha\rho)[b_1\cos\mu z + b_2 sen\,\mu z]\,, \\
&\quad \alpha=\sqrt{\mu^2-\lambda^2}\,,\ \ \lambda^2<\mu^2\,,
\end{aligned}
$$

logo,

$$
\begin{aligned}
u(t,\rho,z) &= u_e(\rho,z) + \sum_{\lambda\mu} e^{-\lambda^2\kappa t}\left\{J_0(\lambda\rho)(a_0+b_0 z)\right. \\
&\quad \left. +I_0(\alpha\rho)[b_1\cos\mu z + b_2 sen\,\mu z]\right\}\,, \\
&\quad \alpha^2+\lambda^2=\mu^2\,,\ \ \lambda^2<\mu^2\,.
\end{aligned}
$$

As condições de contorno são,

$$\begin{aligned}
u(t,a,z) &= f(z) = u_e(a,z) + \sum_{\lambda\mu} e^{-\lambda^2\kappa t}\left\{J_0(\lambda a)(a_0 + b_0 z)\right. \\
&\quad \left. + I_0(\alpha a)[b_1 \cos\mu z + b_2 sen\,\mu z]\right\}\ , \\
u(t,\rho,0) &= g(\rho) = u_e(\rho,0) + \sum_{\lambda\mu} e^{-\lambda^2\kappa t}\left\{J_0(\lambda\rho)a_0\right. \\
&\quad \left. + I_0(\alpha\rho)b_1\right\}\ , \\
u(t,\rho,L) &= h(\rho) = u_e(\rho,L) + \sum_{\lambda\mu} e^{-\lambda^2\kappa t}\left\{J_0(\lambda\rho)(a_0 + b_0 L)\right. \\
&\quad \left. + I_0(\alpha\rho)[b_1 \cos\mu L + b_2 sen\,\mu L]\right\}\ , \\
u(0,\rho,z) &= v(\rho,z) = u_e(\rho,z) + \sum_{\lambda\mu} \left\{J_0(\lambda\rho)(a_0 + b_0 z)\right. \\
&\quad \left. + I_0(\alpha\rho)[b_1 \cos\mu z + b_2 sen\,\mu z]\right\}\ , \\
&\quad \alpha^2 + \lambda^2 = \mu^2\,,\quad \lambda^2 < \mu^2\,.
\end{aligned}$$

Como I_0 não possui raízes, não podemos satisfazer as condições acima.

(c) Escrevemos a solução como,

$$u(t,\rho,z) = u_e(\rho,z) + \sum_{\lambda\mu} e^{-\lambda^2\kappa t} F(\rho,z)\,,$$

com,

$$\begin{aligned}
F(\rho,z) &= [c_1 J_0(\lambda\rho) + c_2 Y_0(\lambda\rho)](a_0 + b_0 z) \\
&\quad + [c_1 J_0(\alpha\rho) + c_2 Y_0(\alpha\rho)][b_1 cosh\mu z + b_2 senh\mu z]\,, \\
&\quad \alpha = \sqrt{\lambda^2 + \mu^2}\,,
\end{aligned}$$

assim, a solução finita em $\rho = 0$ é,

$$\begin{aligned}
u(t,\rho,z) &= u_e(\rho,z) + \sum_{\lambda\mu} e^{-\lambda^2\kappa t}\left\{J_0(\lambda\rho)(a_0 + b_0 z)\right. \\
&\quad \left. + J_0(\alpha\rho)[b_1 cosh\mu z + b_2 senh\mu z]\right\}\ , \\
&\quad \alpha = \sqrt{\lambda^2 + \mu^2}\,.
\end{aligned}$$

As condições de contorno são,

$$
\begin{aligned}
u(t,a,z) &= f(z) = u_e(a,z) + \sum_{\lambda\mu} e^{-\lambda^2\kappa t}\left\{J_0(\lambda a)(a_0 + b_0 z)\right. \\
&\quad \left. + J_0(\alpha a)[b_1 cosh\mu z + b_2 senh\mu z]\right\}\ , \\
u(t,\rho,0) &= g(\rho) = u_e(\rho,0) + \sum_{\lambda\mu} e^{-\lambda^2\kappa t}\left\{J_0(\lambda\rho)a_0\right. \\
&\quad \left. + J_0(\alpha\rho)b_1\right\}\ , \\
u(t,\rho,L) &= h(\rho) = u_e(\rho,L) + \sum_{\lambda\mu} e^{-\lambda^2\kappa t}\left\{J_0(\lambda\rho)(a_0 + b_0 L)\right. \\
&\quad \left. + J_0(\alpha\rho)[b_1 cosh\mu L + b_2 senh\mu L]\right\}\ , \\
u(0,\rho,z) &= v(\rho,z) = u_e(\rho,z) + \sum_{\lambda\mu} \left\{J_0(\lambda\rho)(a_0 + b_0 z)\right. \\
&\quad \left. + J_0(\alpha\rho)[b_1 cosh\mu z + b_2 senh\mu z]\right\}\ , \\
&\quad \alpha = \sqrt{\lambda^2 + \mu^2}\,.
\end{aligned}
$$

Como não podemos expandir as funções em z, a solução nessa forma não é possível.

2. Consideremos a região $0 \le \rho \le a$ e $0 \le z \le L$. Calcule a temperatura $u(t,\rho,z)$ sendo ([7], probl. 6.113 com $f = g = h = 0$),

$$
\begin{aligned}
&\frac{\partial u(t,a,z)}{\partial \rho} = f(z)\,, \\
&u(t,\rho,0) = g(\rho)\,, \\
&u(t,\rho,L) = h(\rho)\,, \\
&u(0,\rho,z) = v(\rho,z)\,.
\end{aligned}
$$

3. Consideremos a região $0 \le \rho \le a$ e $0 \le z \le L$. Calcule a temperatura $u(t,\rho,z)$ sendo,

$$
\begin{aligned}
&u(t,a,z) = f(z)\,, \\
&\frac{\partial u(t,\rho,0)}{\partial z} = g(\rho)\,, \\
&u(t,\rho,L) = h(\rho)\,, \\
&u(0,\rho,z) = v(\rho,z)\,.
\end{aligned}
$$

4. Consideremos a região $0 \le \rho \le a$ e $0 \le z \le L$. Calcule a temperatura $u(t,\rho,z)$ sendo ([7], probl. 6.112 com $f = g = h = 0$),

$$
\begin{aligned}
&u(t,a,z) = f(z)\,,\\
&u(t,\rho,0) = g(\rho)\,,\\
&\frac{\partial u(t,\rho,L)}{\partial z} = h(\rho)\,,\\
&u(0,\rho,z) = v(\rho,z)\,.
\end{aligned}
$$

5. Consideremos a região $0 \le \rho \le a$ e $0 \le z \le L$. Calcule a temperatura $u(t,\rho,z)$ para os seguintes casos:

(a) $u_z(t,\rho,0) = g(\rho)$, $u_z(t,\rho,L) = h(\rho)$, $u(t,a,z) = f(z)$, $u(0,\rho,z) = v(\rho,z)$.

(b) $u_z(t,\rho,0) = g(\rho)$, $u(t,\rho,L) = h(\rho)$, $u_\rho(t,a,z) = f(z)$, $u(0,\rho,z) = v(\rho,z)$.

(c) $u(t,\rho,0) = g(\rho)$, $u_z(t,\rho,L) = h(\rho)$, $u_\rho(t,a,z) = f(z)$, $u(0,\rho,z) = v(\rho,z)$.

(d) $u_z(t,\rho,0) = g(\rho)$, $u_z(t,\rho,L) = h(\rho)$, $u_\rho(t,a,z) = f(z)$, $u(0,\rho,z) = v(\rho,z)$.

6. Consideremos a região $0 \le \rho \le a$ e $0 \le z \le L$. Calcule a temperatura $u(t,\rho,z)$ para os seguintes casos:

(a)

$$
\begin{aligned}
&u_z(t,\rho,0) = -h_1[u(t,\rho,0) - u_0]\,,\;\; u_z(t,\rho,L) = -h_2[u(t,\rho,L) - u_0]\,,\\
&u(t,a,z) = f(z)\,,\;\; u(0,\rho,z) = v(\rho,z)\,.
\end{aligned}
$$

(c)

$$
\begin{aligned}
&u_z(t,\rho,0) = -h_1[u(t,\rho,0) - u_0]\,,\;\; u(t,\rho,L) = h(\rho)\,,\\
&u_\rho(t,a,z) = -h_2[u(t,a,z) - u_0]\,,\;\; u(0,\rho,z) = v(\rho,z)\,.
\end{aligned}
$$

(d)

$$
\begin{aligned}
&u(t,\rho,0) = g(\rho)\,,\;\; u_z(t,\rho,L) = -h_1[u(t,\rho,L) - u_0]\,,\\
&u_\rho(t,a,z) = -h_2[u(t,a,z) - u_0]\,,\;\; u(0,\rho,z) = v(\rho,z)\,.
\end{aligned}
$$

(e)

$$
\begin{aligned}
&u_z(t,\rho,0) = -h_1[u(t,\rho,0) - u_0]\,,\;\; u_z(t,\rho,L) = -h_2[u(t,\rho,L) - u_0]\,,\\
&u_\rho(t,a,z) = -h_3[u(t,a,z) - u_0]\,,\;\; u(0,\rho,z) = v(\rho,z)\,.
\end{aligned}
$$

7. Consideremos a região $0 \le \rho \le a$ e $0 \le z \le L$. Calcule a temperatura $u(t,\rho,z)$ com,

$$
\begin{aligned}
&u(t,a,z) = f(t,z)\,, \quad u(t,\rho,0) = g(t,\rho)\,,\\
&u(t,\rho,L) = h(t,\rho)\,, \quad u(0,\rho,z) = v(\rho,z)\,.
\end{aligned}
$$

8. Consideremos um cilindro infinito de raio a. Calcule a temperatura $u(t,\rho,z)$ sendo,

$$
\begin{aligned}
&u(t,a,z) = f(t,z)\,,\\
&u(0,\rho,z) = g(\rho,z)\,.
\end{aligned}
$$

9. Considere a região $a \le \rho \le b$ e $0 \le z \le L$. Calcule a temperatura $u(t,\rho,z)$ sendo ([7], probl. 6.118 com $f = g = h = v = 0$),

$$
\begin{aligned}
&u(t,a,z) = f(z)\,, \quad u(t,b,z) = g(z)\,,\\
&u(t,\rho,0) = h(\rho)\,, \quad u(t,\rho,L) = v(\rho)\,,\\
&u(0,\rho,z) = w(\rho,z)\,.
\end{aligned}
$$

10. Consideremos a região $a \le \rho \le b$ e $0 \le z \le L$. Calcule a temperatura $u(t,\rho,z)$ sendo ([7], probl. 6.118 com $f = g = h = v = 0$),

$$
\begin{aligned}
&u(t,a,z) = f(t,z)\,, \quad u(t,b,z) = g(t,z)\,,\\
&u(t,\rho,0) = h(t,\rho)\,, \quad u(t,\rho,L) = v(t,\rho)\,,\\
&u(0,\rho,z) = w(\rho,z)\,.
\end{aligned}
$$

11. Consideremos a região $a \le \rho \le b$ e $0 \le z \le L$. Calcule a temperatura $u(t,\rho,z)$ sendo ([7], probl. 6.120(a) com $f = g = h = v = 0$),

$$
\begin{aligned}
&u(t,a,z) = f(t,z)\,,\\
&u(t,b,z) = g(t,z)\,,\\
&\frac{\partial u(t,\rho,0)}{\partial z} = h(t,\rho)\,,\\
&\frac{\partial u(t,\rho,L)}{\partial z} = v(t,\rho)\,,\\
&u(0,\rho,z) = w(\rho,z)\,.
\end{aligned}
$$

12. Consideremos a região $a \le \rho \le b$ e $0 \le z \le L$. Calcule a temperatura $u(t,\rho,z)$ sendo ([7], probl. 6.120(b) com $f = g = h = v = 0$),

$$\begin{aligned}
\frac{\partial u(t,a,z)}{\partial \rho} &= f(t,z)\,, \\
\frac{\partial u(t,b,z)}{\partial \rho} &= g(t,z)\,, \\
u(t,\rho,0) &= h(t,\rho)\,, \\
u(t,\rho,L) &= v(t,\rho)\,, \\
u(0,\rho,z) &= w(\rho,z)\,.
\end{aligned}$$

13. Consideremos a região $a \leq \rho \leq b$ e $0 \leq z \leq L$. Calcule a temperatura $u(t,\rho,z)$ sendo ([7], probl. 6.120(c) com $f = g = h = v = 0$),

$$\begin{aligned}
\frac{\partial u(t,a,z)}{\partial \rho} &= f(t,z)\,, \\
\frac{\partial u(t,b,z)}{\partial \rho} &= g(t,z)\,, \\
\frac{\partial u(t,\rho,0)}{\partial z} &= h(t,\rho)\,, \\
\frac{\partial u(t,\rho,L)}{\partial z} &= v(t,\rho)\,, \\
u(0,\rho,z) &= w(\rho,z)\,.
\end{aligned}$$

14. Consideremos uma casca cilíndrica infinita de raio interno a e raio externo b. Calcule a temperatura $u(t,\rho,z)$ sendo,

$$\begin{aligned}
u(t,a,z) &= f(t,z)\,, \\
u(t,b,z) &= g(t,z)\,, \\
u(0,\rho,z) &= h(\rho,z)\,.
\end{aligned}$$

15. Considerando,

$$\begin{aligned}
&0 \leq \rho \leq a\,, \quad 0 \leq z \leq L\,, \\
&u(t,a,z) = \mu(z)\,, \quad u(t,\rho,0) = \nu_1(\rho)\,, \\
&u(t,\rho,L) = \nu_2(\rho)\,, \quad u(0,\rho,z) = g(\rho,z)\,,
\end{aligned}$$

calcule $u(t,\rho,z)$ para:

(a)

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + F(\rho, z)\,,$$

(b)

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + \alpha^2\,,$$

com α constante.

(c)

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + \alpha^2 u\,,$$

com α constante.

16. Considerando,

$$\begin{aligned}
&0 \le \rho \le a\,, \;\; 0 \le z \le L\,,\\
&u(t,a,z) = \mu(t,z)\,, \;\; u(t,\rho,0) = \nu_1(t,\rho)\,,\\
&u(t,\rho,L) = \nu_2(t,\rho)\,, \;\; u(0,\rho,z) = g(\rho,z)\,,
\end{aligned}$$

calcule $u(t,\rho,z)$ para:

(a)

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + F(t, \rho, z)\,,$$

(b)

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u\,,$$

(c)

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + \alpha^2\,,$$

com α constante,

(d)

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + \alpha^2 u\,,$$

com α constante.

17. Considerando,

$$
\begin{aligned}
&a \le \rho \le b\,, \quad 0 \le z \le L\,,\\
&u(t,a,z) = \mu_1(z)\,, \quad u(t,b,z) = \mu_2(z)\,,\\
&u(t,\rho,0) = \nu_1(\rho)\,, \quad u(t,\rho,L) = \nu_2(\rho)\,,\\
&u(0,\rho,z) = g(\rho,z)\,.
\end{aligned}
$$

calcule $u(t,\rho,z)$ para:

(a)

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + F(\rho,z)\,,$$

(b)

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + \alpha^2\,,$$

com α constante,

(c)

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + \alpha^2 u\,,$$

com α constante.

18. Considerando,

$$
\begin{aligned}
&a \le \rho \le b\,, \quad 0 \le z \le L\,,\\
&u(t,a,z) = \mu_1(t,z)\,, \quad u(t,b,z) = \mu_2(t,z)\,,\\
&u(t,\rho,0) = \nu_1(t,\rho)\,, \quad u(t,\rho,L) = \nu_2(t,\rho)\,,\\
&u(0,\rho,z) = g(\rho,z)\,.
\end{aligned}
$$

calcule $u(t,\rho,z)$ para:

(a)

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + F(t,\rho,z)\,,$$

(b)

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u\,,$$

(c)

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + \alpha^2 \,,$$

com α constante,

(d)

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + \alpha^2 u \,,$$

com α constante.

5 Considerando $F(\rho, \varphi)$

Temos agora a equação,

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial F}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 F}{\partial \varphi^2} + \lambda^2 F = 0 \,. \quad (17)$$

Substituindo $F(\rho, \varphi) = R(\rho)\Phi(\varphi)$ obtemos,

$$\rho^2 \frac{R''}{R} + \rho \frac{R'}{R} + \frac{\Phi''}{\Phi} + \lambda^2 \rho^2 = 0 \,,$$

ou,

$$\rho^2 \frac{R''}{R} + \rho \frac{R'}{R} + \lambda^2 \rho^2 = -\frac{\Phi''}{\Phi} \equiv \mu^2 \,.$$

A equação para R é,

$$\rho^2 R'' + \rho R' + (\lambda^2 \rho^2 - \mu^2) R = 0 \,, \quad (18)$$

com solução,

$$R(\rho) = c_1 J_\mu(\lambda \rho) + c_2 Y_\mu(\lambda \rho) \,. \quad (19)$$

A equação para Φ é,

$$\Phi'' + \mu^2 \Phi = 0 \,, \quad (20)$$

com solução,

$$\Phi(\varphi) = b_1 \cos \mu\varphi + b_2 \mathrm{sen}\, \mu\varphi \,. \quad (21)$$

A solução $u(t, \rho, \varphi)$ é então,

$$
\begin{aligned}
u(t,\rho,\varphi) &= u_e(\rho,\varphi) + \sum_{\lambda\mu} \exp(-\lambda^2 \kappa t) \times \\
&+ \times \left[c_1 J_\mu(\lambda\rho) + c_2 Y_\mu(\lambda\rho)\right]\left[b_1 \cos\mu\varphi + b_2 \mathrm{sen}\,\mu\varphi\right].
\end{aligned}
$$

6 Problemas

1. Consideremos a região $0 \leq \rho \leq a$. Calcule a temperatura $u(t,\rho,\varphi)$ sendo $u(t,a,\varphi) = f(\varphi)$, $u(0,\rho,\varphi) = g(\rho,\varphi)$.

A solução finita em $\rho = 0$ é,

$$
\begin{aligned}
u(t,\rho,\varphi) &= u_e(\rho,\varphi) + \sum_{\lambda\mu} \exp(-\lambda^2 \kappa t) \times \\
& \times J_\mu(\lambda\rho)\left[b_1 \cos\mu\varphi + b_2 sen\,\mu\varphi\right]
\end{aligned}
$$

Escrevendo as condições de contorno,

$$
\begin{aligned}
u(t,a,\varphi) &= f(\varphi) = u_e(a,\varphi) + \sum_{\lambda\mu} \exp(-\lambda^2 \kappa t) \times \\
& \times J_\mu(\lambda a)\left[b_1 \cos\mu\varphi + b_2 sen\,\mu\varphi\right], \\
u(0,\rho,\varphi) &= g(\rho,\varphi) = u_e(\rho,\varphi) + \\
& + \sum_{\lambda\mu} J_\mu(\lambda\rho)\left[b_1 \cos\mu\varphi + b_2 sen\,\mu\varphi\right].
\end{aligned}
$$

Satisfazemos as equações acima escolhendo,

$$
J_\mu(\lambda_{\mu j} a) = 0\,, \quad j = 1, 2, \ldots,
$$

assim,

$$
\begin{aligned}
f(\varphi) &= u_e(a,\varphi) \\
g(\rho,\varphi) &= u_e(\rho,\varphi) + \sum_{j\mu} J_\mu(\lambda_{j\mu}\rho)\left[b_{1\mu} \cos\mu\varphi + b_{2\mu} sen\,\mu\varphi\right].
\end{aligned}
$$

A segunda equação acima é uma expansão em série de Fourier e uma série de funções de Bessel. Temos então,

$$\sum_j J_0(\lambda_{0j}\rho) b_{10} = \frac{1}{2\pi}\int_{-\pi}^{\pi}[g(\rho,x) - u_e(\rho,x)]\,dx\,,$$
$$\sum_j J_\mu(\lambda_{\mu j}\rho) b_{1\mu} = \frac{1}{\pi}\int_{-\pi}^{\pi}[g(\rho,x) - u_e(\rho,x)]\cos\mu x\,dx\,,$$
$$\sum_j J_\mu(\lambda_{\mu j}\rho) b_{2\mu} = \frac{1}{\pi}\int_{-\pi}^{\pi}[g(\rho,x) - u_e(\rho,x)] sen\,\mu x\,dx\,.$$

As expressões acima são séries de funções de Bessel, portanto,

$$b_{10} = \frac{2}{a^2 J_1^2(\lambda_{0j}a)}\int_0^a \rho J_0(\lambda_{0j}\rho)d\rho\frac{1}{2\pi}\int_{-\pi}^{\pi}[g(\rho,x) - u_e(\rho,x)]\,dx\,,$$
$$b_{1\mu} = \frac{2}{a^2 J_{\mu+1}^2(\lambda_{\mu j}a)}\int_0^a \rho J_\mu(\lambda_{\mu j}\rho)d\rho\frac{1}{\pi}\int_{-\pi}^{\pi}[g(\rho,x) - u_e(\rho,x)]\cos\mu x\,dx\,,$$
$$b_{2\mu} = \frac{2}{a^2 J_{\mu+1}^2(\lambda_{\mu j}a)}\int_0^a \rho J_\mu(\lambda_{\mu j}\rho)d\rho\frac{1}{\pi}\int_{-\pi}^{\pi}[g(\rho,x) - u_e(\rho,x)] sen\,\mu x\,dx\,.$$

Obtemos assim a seguinte série,

$$\begin{aligned} g(\rho,\varphi) \;=\;& u_e(\rho,\varphi) + \sum_j J_0(\lambda_{0j}\rho)\times \\ & \times\frac{2}{a^2 J_1^2(\lambda_{0j}a)}\int_0^a \rho J_0(\lambda_{0j}\rho)d\rho\frac{1}{2\pi}\int_{-\pi}^{\pi}[g(\rho,x) - u_e(\rho,x)]\,dx \\ & + \sum_{\mu j} J_\mu(\lambda_{\mu j}\rho)\cos\mu\varphi\times \\ & \times\frac{2}{a^2 J_{\mu+1}^2(\lambda_{\mu j}a)}\int_0^a \rho J_\mu(\lambda_{\mu j}\rho)d\rho\frac{1}{\pi}\int_{-\pi}^{\pi}[g(\rho,x) - u_e(\rho,x)]\cos\mu x\,dx \\ & + \sum_{\mu j} J_\mu(\lambda_{\mu j}\rho) sen\,\mu\varphi\times \\ & \times\frac{2}{a^2 J_{\mu+1}^2(\lambda_{\mu j}a)}\int_0^a \rho J_\mu(\lambda_{\mu j}\rho)d\rho\frac{1}{\pi}\int_{-\pi}^{\pi}[g(\rho,x) - u_e(\rho,x)] sen\,\mu x\,dx\,. \end{aligned}$$

A solução é, dessa forma,

$$
\begin{aligned}
u(t,\rho,\varphi) &= u_e(\rho,\varphi) + \sum_j \exp(-\lambda_{0j}^2\kappa t) J_0(\lambda_{0j}\rho) \times \\
&\times \frac{2}{a^2 J_1^2(\lambda_{0j}a)} \int_0^a \rho J_0(\lambda_{0j}\rho) d\rho \frac{1}{2\pi} \int_{-\pi}^{\pi} [g(\rho,x) - u_e(\rho,x)]\, dx \\
&+ \sum_{\mu j} \exp(-\lambda_{\mu j}^2 \kappa t) J_\mu(\lambda_{\mu j}\rho) \cos\mu\varphi \times \\
&\times \frac{2}{a^2 J_{\mu+1}^2(\lambda_{\mu j}a)} \int_0^a \rho J_\mu(\lambda_{\mu j}\rho) d\rho \frac{1}{\pi} \int_{-\pi}^{\pi} [g(\rho,x) - u_e(\rho,x)] \cos\mu x\, dx \\
&+ \sum_{\mu j} \exp(-\lambda_{\mu j}^2 \kappa t) J_\mu(\lambda_{\mu j}\rho) sen\,\mu\varphi \times \\
&\times \frac{2}{a^2 J_{\mu+1}^2(\lambda_{\mu j}a)} \int_0^a \rho J_\mu(\lambda_{\mu j}\rho) d\rho \frac{1}{\pi} \int_{-\pi}^{\pi} [g(\rho,x) - u_e(\rho,x)] sen\,\mu x\, dx\,.
\end{aligned}
$$

Se $g(\rho,x) = g_0$ constante e $u_e(\rho,x) = u_e$ também constante,

$$
\begin{aligned}
u(t,\rho,\varphi) &= u_e + \sum_j \exp(-\lambda_{0j}^2\kappa t) J_0(\lambda_{0j}\rho) \times \\
&\times \frac{2}{a^2 J_1^2(\lambda_{0j}a)} \int_0^a \rho J_0(\lambda_{0j}\rho) d\rho\, (g_0 - u_e)\,, \\
&= u_e + \frac{2(g_0 - u_e)}{a} \sum_j \exp(-\lambda_{0j}^2\kappa t) \frac{J_0(\lambda_{0j}\rho)}{\lambda_{0j} J_1(\lambda_{0j}a)}\,.
\end{aligned}
$$

Em $t = 0$,

$$
u(0,\rho,\varphi) = g_0 = u_e + \frac{2(g_0 - u_e)}{a} \sum_j \frac{J_0(\lambda_{0j}\rho)}{\lambda_{0j} J_1(\lambda_{0j}a)} = g_0\,,
$$

como esperado.

2. Consideremos a região $a \le \rho \le b$. Calcule a temperatura $u(t,\rho,\varphi)$ sendo $u(t,a,\varphi) = f(\varphi)$, $u(t,b,\varphi) = g(\varphi)$, $u(0,\rho,\varphi) = h(\rho,\varphi)$.

A solução é,

$$
\begin{aligned}
u(t,\rho,\varphi) &= u_e(\rho,\varphi) + \sum_{\lambda\mu} \exp(-\lambda^2\kappa t) \times \\
&\times [c_{1\lambda} J_\mu(\lambda\rho) + c_{2\lambda} Y_\mu(\lambda\rho)][b_{1\mu}\cos\mu\varphi + b_{2\mu} sen\,\mu\varphi]\,,
\end{aligned}
$$

e as condições de contorno,

$$\begin{aligned}
u(t,a,\varphi) &= f(\varphi) = u_e(a,\varphi) + \sum_{\lambda\mu} \exp(-\lambda^2\kappa t)\times \\
&\times[c_{1\lambda}J_\mu(\lambda a) + c_{2\lambda}Y_\mu(\lambda a)][b_{1\mu}\cos\mu\varphi + b_{2\mu} sen\,\mu\varphi]\,, \\
u(t,b,\varphi) &= g(\varphi) = u_e(b,\varphi) + \sum_{\lambda\mu} \exp(-\lambda^2\kappa t)\times \\
&\times[c_{1\lambda}J_\mu(\lambda b) + c_{2\lambda}Y_\mu(\lambda b)][b_{1\mu}\cos\mu\varphi + b_{2\mu} sen\,\mu\varphi]\,, \\
u(0,\rho,\varphi) &= h(\rho,\varphi) = u_e(\rho,\varphi) \\
&+ \sum_{\lambda\mu}[c_{1\lambda}J_\mu(\lambda\rho) + c_{2\lambda}Y_\mu(\lambda\rho)][b_{1\mu}\cos\mu\varphi + b_{2\mu} sen\,\mu\varphi]\,.
\end{aligned}$$

Escolhemos portanto,

$$\begin{aligned}
c_{1\lambda}J_\mu(\lambda a) + c_{2\lambda}Y_\mu(\lambda a) = 0\,, \\
c_{1\lambda}J_\mu(\lambda b) + c_{2\lambda}Y_\mu(\lambda b) = 0\,.
\end{aligned}$$

Os valores de λ são determinados assim por,

$$J_\mu(\lambda_{\mu j}a)Y_\mu(\lambda_{\mu j}b) - J_\mu(\lambda_{\mu j}b)Y_\mu(\lambda_{\mu j}a) = 0\,, \quad j = 1,2,\ldots$$

ou,

$$U_\mu(\lambda_{\mu j}b) = 0\,, \quad j = 1,2,\ldots$$

Podemos escrever então,

$$c_{2\lambda} = -c_{1\lambda}\frac{J_\mu(\lambda a)}{Y_\mu(\lambda a)}\,.$$

Temos então, fazendo $c_{1\lambda} = 1$,

$$\begin{aligned}
u(t,\rho,\varphi) &= u_e(\rho,\varphi) + \sum_{\mu j}\exp(-\lambda_{\mu j}^2\kappa t)\times \\
&\times\frac{U_\mu(\lambda_{\mu j}\rho)}{Y_\mu(\lambda_{\mu j}a)}[b_{1\mu}\cos\mu\varphi + b_{2\mu} sen\,\mu\varphi]\,,
\end{aligned}$$

$$\begin{aligned}
&f(\varphi) = u_e(a,\varphi)\,, \\
&g(\varphi) = u_e(b,\varphi)\,, \\
&h(\rho,\varphi) = u_e(\rho,\varphi) + \sum_{\mu j}\frac{U_\mu(\lambda_{\mu j}\rho)}{Y_\mu(\lambda_{\mu j}a)}[b_{1\mu}\cos\mu\varphi + b_{2\mu} sen\,\mu\varphi]\,.
\end{aligned}$$

A última expressão é uma série de Fourier, logo,

$$\sum_j \frac{U_0(\lambda_{0j},\rho)}{Y_0(\lambda_{0j}a)}b_{10} = \frac{1}{2\pi}\int_{-\pi}^{\pi}[h(\rho,x) - u_e(\rho,x)]\,dx\,,$$

$$\sum_j \frac{U_\mu(\lambda_{\mu j}\rho)}{Y_\mu(\lambda_{\mu j}a)}b_{1\mu} = \frac{1}{\pi}\int_{-\pi}^{\pi}[h(\rho,x) - u_e(\rho,x)]\cos\mu x\,dx\,,$$

$$\sum_j \frac{U_\mu(\lambda_{\mu j}\rho)}{Y_\mu(\lambda_{\mu j}a)}b_{2\mu} = \frac{1}{\pi}\int_{-\pi}^{\pi}[h(\rho,x) - u_e(\rho,x)] sen\,\mu x\,dx\,.$$

As expressões acima são séries de funções de Bessel, portanto,

$$b_{10} = \frac{Y_0(\lambda_{0j}a)}{\displaystyle\int_a^b \rho[U_0(\lambda_{0j}\rho)]^2 d\rho}\int_a^b yU_0(\lambda_{0j}y)dy\frac{1}{2\pi}\int_{-\pi}^{\pi}[h(y,x) - u_e(y,x)]\,dx\,,$$

$$b_{1\mu} = \frac{Y_\mu(\lambda_{\mu j}a)}{\displaystyle\int_a^b \rho[U_\mu(\lambda_{\mu j}\rho)]^2 d\rho}\int_a^b yU_\mu(\lambda_{\mu j}y)dy\frac{1}{\pi}\int_{-\pi}^{\pi}[h(y,x) - u_e(y,x)]\cos\mu x\,dx\,,$$

$$b_{2\mu} = \frac{Y_\mu(\lambda_{\mu j}a)}{\displaystyle\int_a^b \rho[U_\mu(\lambda_{\mu j}\rho)]^2 d\rho}\int_a^b yU_\mu(\lambda_{\mu j}y)dy\frac{1}{\pi}\int_{-\pi}^{\pi}[h(y,x) - u_e(y,x)] sen\,\mu x\,dx\,.$$

A solução é assim,

$$\begin{aligned}
u(t,\rho,\varphi) &= u_e(\rho,\varphi) + \sum_j \exp(-\lambda_{0j}^2\kappa t)\frac{U_0(\lambda_{0j}\rho)}{\displaystyle\int_a^b \rho[U_0(\lambda_{0j}\rho)]^2 d\rho} \times \\
&\times \int_a^b yU_0(\lambda_{0j}y)dy \frac{1}{2\pi}\int_{-\pi}^{\pi}[h(y,x)-u_e(y,x)]\,dx \\
&+ \sum_{\mu j} \exp(-\lambda_{\mu j}^2\kappa t)\frac{U_\mu(\lambda_{\mu j}\rho)}{\displaystyle\int_a^b \rho[U_\mu(\lambda_{\mu j}\rho)]^2 d\rho}\cos\mu\varphi \times \\
&\times \int_a^b yU_\mu(\lambda_{\mu j}y)dy \frac{1}{\pi}\int_{-\pi}^{\pi}[h(y,x)-u_e(y,x)]\cos\mu x\,dx \\
&+ \sum_{\mu j} \exp(-\lambda_{\mu j}^2\kappa t)\frac{U_\mu(\lambda_{\mu j}\rho)}{\displaystyle\int_a^b \rho[U_\mu(\lambda_{\mu j}\rho)]^2 d\rho} sen\,\mu\varphi \times \\
&\times \int_a^b yU_\mu(\lambda_{\mu j}y)dy \frac{1}{\pi}\int_{-\pi}^{\pi}[h(y,x)-u_e(y,x)] sen\,\mu x\,dx\,.
\end{aligned}$$

Se $h(\rho,\varphi) = h_0$ *constante e* $u_e(\rho,\varphi) = u_e$ *constante,*

$$\begin{aligned}
u(t,\rho,\varphi) &= u_e + \sum_j \exp(-\lambda_{0j}^2\kappa t)\frac{U_0(\lambda_{0j}\rho)}{\displaystyle\int_a^b \rho[U_0(\lambda_{0j}\rho)]^2 d\rho} \times \\
&\times \int_a^b yU_0(\lambda_{0j}y)dy[h_0 - u_e]\,, \\
&= u_e + (h_0 - u_e)\sum_j \exp(-\lambda_{0j}^2\kappa t)U_0(\lambda_{0j}\rho)\frac{\displaystyle\int_a^b yU_0(\lambda_{0j}y)dy}{\displaystyle\int_a^b \rho[U_0(\lambda_{0j}\rho)]^2 d\rho}\,.
\end{aligned}$$

Em $t = 0$ *temos* $u(0,\rho,\varphi) = g_0$ *como esperado.*

3. Consideremos a região $0 \le \rho \le a$, $0 \le \varphi \le \beta$. Calcule a temperatura $u(t,\rho,\varphi)$, com ([7], probl. 2.76 para o caso estacionário com $u_1 = u_2 = 0$),

$$\begin{aligned}
&u(t,a,\varphi) = f(\varphi)\,, \\
&u(t,\rho,0) = u_1(\rho)\,, \quad u(t,\rho,\beta) = u_2(\rho)\,, \\
&u(0,\rho,\varphi) = g(\rho,\varphi)\,.
\end{aligned}$$

4. Calcule $u(t,\rho,\varphi)$ para,
(a)
$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + F(\rho,\varphi)\,,$$
(b)
$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + \alpha^2\,,$$
com α constante,
(c)
$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + \alpha^2 u\,,$$
com α constante e,

$$0 \le \rho \le a\,,\;\; 0 \le \varphi \le 2\pi\,,$$
$$u(t,a,\varphi) = \mu(\varphi)\,,\;\; u(0,\rho,\varphi) = g(\rho,\varphi)\,.$$

5. Calcule $u(t,\rho,\varphi)$ para,
(a)
$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u\,,$$
(b)
$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + \alpha^2\,,$$
com α constante,
(c)
$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + \alpha^2 u\,,$$
com α constante,
(d)
$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + F(t,\rho,\varphi)\,,$$
e,

$$0 \le \rho \le a\,,\;\; 0 \le \varphi \le 2\pi\,,$$
$$u(t,a,\varphi) = f(t,\varphi)\,,\;\; u(0,\rho,\varphi) = g(\rho,\varphi)\,.$$

6. Calcule $u(t,\rho,\varphi)$ para,
(a)
$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + F(\rho,\varphi)\,,$$

(b)

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + \alpha^2 \,,$$

com α constante,

(c)

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + \alpha^2 u \,,$$

com α constante e,

$$\begin{aligned}
&a \leq \rho \leq b \,, \quad 0 \leq \varphi \leq 2\pi \,, \\
&u(t,a,\varphi) = f(\varphi) \,, \quad u(t,b,\varphi) = g(\varphi) \,, \\
&u(0,\rho,\varphi) = h(\rho,\varphi) \,.
\end{aligned}$$

7. Calcule $u(t,\rho,\varphi)$ para,

(a)

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + F(t,\rho,\varphi) \,,$$

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u \,,$$

(b)

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + \alpha^2 \,,$$

com α constante,

(c)

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + \alpha^2 u \,,$$

com α constante e,

$$\begin{aligned}
&a \leq \rho \leq b \,, \quad 0 \leq \varphi \leq 2\pi \,, \\
&u(t,a,\varphi) = f(t,\varphi) \,, \quad u(t,b,\varphi) = g(t,\varphi) \,, \\
&u(0,\rho,\varphi) = h(\rho,\varphi) \,.
\end{aligned}$$

7 Considerando $F(\rho,\varphi,z)$

A equação para F é agora,

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial F}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 F}{\partial \varphi^2} + \frac{\partial^2 F}{\partial z^2} + \lambda^2 F = 0 \,. \qquad (22)$$

Substituindo $F(\rho, \varphi) = R(\rho)\Phi(\varphi)Z(z)$ obtemos,

$$\frac{1}{\rho}\Phi Z(\rho R')' + \frac{1}{\rho^2}RZ\Phi'' + R\Phi Z'' + \lambda^2 R\Phi Z = 0\,,$$

ou,

$$\frac{1}{\rho R}(\rho R')' + \frac{1}{\rho^2}\frac{\Phi''}{\Phi} + \lambda^2 = -\frac{Z''}{Z} = +\alpha^2\,. \tag{23}$$

Temos assim,

$$\frac{\rho}{R}(\rho R')' + \frac{\Phi''}{\Phi} + (\lambda^2 - \alpha^2)\rho^2 = 0\,,$$

ou,

$$\frac{\rho}{R}(\rho R')' + (\lambda^2 - \alpha^2)\rho^2 = -\frac{\Phi''}{\Phi} = \mu^2\,.$$

A equação para R é então,

$$\rho^2 R'' + \rho R' + [(\lambda^2 - \alpha^2)\rho^2 - \mu^2]R = 0\,,$$

com solução,

$$\begin{aligned} R(\rho) &= c_1 J_\mu(\beta\rho) + c_2 Y_\mu(\beta\rho)\,, \\ &\ \lambda^2 - \alpha^2 = \beta^2\,,\ \ \lambda^2 > \alpha^2\,, \\ R(\rho) &= c_1 I_\mu(\beta\rho) + c_2 K_\mu(\beta\rho)\,, \\ &\ \alpha^2 - \lambda^2 = \beta^2\,,\ \ \lambda^2 < \alpha^2\,. \end{aligned}$$

Para Z obtemos,

$$Z'' + \alpha^2 Z = 0\,,$$

com solução,

$$Z(z) = d_1 \cos\alpha z + d_2 \mathrm{sen}\,\alpha z\,.$$

A equação para Φ é,

$$\Phi'' + \mu^2\Phi = 0\,,$$

com solução,

$$\Phi(\varphi) = b_1 \cos\mu\varphi + b_2 \mathrm{sen}\,\mu\varphi\,.$$

Consideramos agora a outra possibilidade para (23),

$$\frac{1}{\rho R}(\rho R')' + \frac{1}{\rho^2}\frac{\Phi''}{\Phi} + \lambda^2 = -\frac{Z''}{Z} = -\alpha^2\,, \tag{24}$$

portanto,

$$\frac{1}{\rho R}(\rho R')' + \frac{1}{\rho^2}\frac{\Phi''}{\Phi} + \lambda^2 + \alpha^2 = 0\,,$$

ou,

$$\frac{\rho}{R}(\rho R')' + (\lambda^2 + \alpha^2)\rho^2 = -\frac{\Phi''}{\Phi} = \mu^2\,.$$

A equação para R é agora,

$$\rho^2 R'' + \rho R' + [(\lambda^2 + \alpha^2)\rho^2 - \mu^2]R = 0\,,$$

com solução,

$$\begin{aligned} R(\rho) &= c_1 J_\mu(\beta\rho) + c_2 Y_\mu(\beta\rho)\,, \\ & \lambda^2 + \alpha^2 = \beta^2\,,\;\; \lambda^2 > \alpha^2\,. \end{aligned}$$

A equação para Z é,

$$Z'' - \alpha^2 Z = 0\,,$$

com solução,

$$Z(z) = d_1 \cosh\alpha z + d_2 \operatorname{senh}\alpha z\,.$$

Para Φ obtemos,

$$\frac{\Phi''}{\Phi} + \mu^2 = 0\,,$$

com solução,

$$\Phi(\varphi) = b_1 \cos\mu\varphi + b_2 \operatorname{sen}\mu\varphi\,.$$

8 Problemas

1. Consideremos a região $0 \leq \rho \leq a$, $0 \leq z \leq L$. Calcule a temperatura $u(t, \rho, \varphi, z)$ sendo ([7], probl. 6.116 com $f = g = h = 0$),

$$u(t, a, \varphi, z) = f(\varphi, z)\,, \quad u(t, \rho, \varphi, 0) = g(\rho, \varphi)\,,$$
$$u(t, \rho, \varphi, L) = h(\rho, \varphi)\,, \quad u(0, \rho, \varphi, z) = w(\rho, \varphi, z)\,.$$

Escrevemos a solução como,

$$\begin{aligned} u(t, \rho, \varphi, z) &= u_e(\rho, \varphi, z) + \sum_{\beta\alpha\mu} \exp(-\lambda^2 \kappa t) J_\mu(\beta\rho) \times \\ &\times [d_{1\alpha} \cos \alpha z + d_{2\alpha} sen\, \alpha z] \times \\ &\times [b_{1\mu} \cos \mu\varphi + b_{2\mu} sen\, \mu\varphi]\,, \\ &\lambda^2 - \alpha^2 = \beta^2\,, \quad \lambda^2 > \alpha^2\,, \end{aligned}$$

em que $u_e(\rho, \varphi, z)$ é solução da equação de Laplace, como antes. As condições de contorno ficam,

$$\begin{aligned} u(t, a, \varphi, z) &= f(\varphi, z) = u_e(a, \varphi, z) \\ &+ \sum_{\beta\alpha\mu} \exp(-\lambda^2 \kappa t) J_\mu(\beta a) \times \\ &\times [d_{1\alpha} \cos \alpha z + d_{2\alpha} sen\, \alpha z] \times \\ &\times [b_{1\mu} \cos \mu\varphi + b_{2\mu} sen\, \mu\varphi]\,, \\ u(t, \rho, \varphi, 0) &= g(\rho, \varphi) = u_e(\rho, \varphi, 0) \\ &+ \sum_{\beta\alpha\mu} \exp(-\lambda^2 \kappa t) J_\mu(\beta\rho) d_{1\alpha} \times \\ &\times [b_{1\mu} \cos \mu\varphi + b_{2\mu} sen\, \mu\varphi]\,, \\ u(t, \rho, \varphi, L) &= h(\rho, \varphi) = u_e(\rho, \varphi, L) \\ &+ \sum_{\beta\alpha\mu} \exp(-\lambda^2 \kappa t) J_\mu(\beta\rho) \times \\ &\times [d_{1\alpha} \cos \alpha L + d_{2\alpha} sen\, \alpha L] \times \\ &\times [b_{1\mu} \cos \mu\varphi + b_{2\mu} sen\, \mu\varphi]\,, \\ u(0, \rho, \varphi, z) &= w(\rho, \varphi, z) = u_e(\rho, \varphi, z) + \sum_{\beta\alpha\mu} J_\mu(\beta\rho) \times \\ &\times [d_{1\alpha} \cos \alpha z + d_{2\alpha} sen\, \alpha z] \times \\ &\times [b_{1\mu} \cos \mu\varphi + b_{2\mu} sen\, \mu\varphi]\,. \end{aligned}$$

Satisfazemos as equações acima escolhendo,

$$
\begin{aligned}
&J_\mu(\beta_{\mu j}a) = 0\,, \quad \mu = 0, 1, 2, \ldots, \quad j = 1, 2, \ldots \\
&d_{1\alpha} = 0\,, \\
&\alpha_k L = k\pi\,, \quad k = 1, 2, \ldots
\end{aligned}
$$

Portanto, escolhendo $d_{2\alpha} = 1$*,*

$$
\begin{aligned}
f(\varphi, z) &= u_e(a, \varphi, z) \\
g(\rho, \varphi) &= u_e(\rho, \varphi, 0) \\
h(\rho, \varphi) &= u_e(\rho, \varphi, L) \\
w(\rho, \varphi, z) &= u_e(\rho, \varphi, z) + \sum_{jk\mu} J_\mu(\beta_{j\mu}\rho) sen\,(k\pi z/L) \times \\
&\times [b_{1\mu} \cos\mu\varphi + b_{2\mu} sen\,\mu\varphi]\,, \\
&\lambda^2 = \alpha^2 + \beta^2\,.
\end{aligned}
$$

A solução fica assim na forma,

$$
\begin{aligned}
u(t, \rho, \varphi, z) &= u_e(\rho, \varphi, z) + \sum_{\mu jk} \exp\{-[(k\pi/L)^2 + \beta_{\mu j}^2]\kappa t\} J_\mu(\beta_{\mu j}\rho) \times \\
&\times sen\,(k\pi z/L)[b_{1\mu} \cos\mu\varphi + b_{2\mu} sen\,\mu\varphi]\,, \\
&\lambda^2 - \alpha^2 = \beta^2\,, \quad \lambda^2 > \alpha^2\,,
\end{aligned}
$$

A condição para w *é uma série de Fourier de senos, uma série de Fourier usual e uma série de funções de Bessel. Considerando primeiro a série de Fourier temos,*

$$
\begin{aligned}
\sum_{jk} J_0(\beta_{0j}\rho) sen\,(k\pi z/L) b_{10} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} [w(\rho, \varphi, z) - u_e(\rho, \varphi, z)] d\varphi\,, \\
\sum_{jk} J_\mu(\beta_{\mu j}\rho) sen\,(k\pi z/L) b_{1\mu} &= \frac{1}{\pi} \int_{-\pi}^{\pi} [w(\rho, \varphi, z) - u_e(\rho, \varphi, z)] \cos\mu\varphi\, d\varphi\,, \\
\sum_{jk} J_\mu(\beta_{\mu j}\rho) sen\,(k\pi z/L) b_{2\mu} &= \frac{1}{\pi} \int_{-\pi}^{\pi} [w(\rho, \varphi, z) - u_e(\rho, \varphi, z)] sen\,\mu\varphi\, d\varphi\,.
\end{aligned}
$$

Considerando agora a série de Fourier de senos,

$$\sum_j J_0(\beta_{0j}\rho) b_{10} = \frac{2}{L}\int_0^L sen\,(k\pi z/L)\,dz \frac{1}{2\pi}\int_{-\pi}^{\pi}[w(\rho,\varphi,z) - u_e(\rho,\varphi,z)]d\varphi\,,$$

$$\sum_j J_\mu(\beta_{\mu j}\rho) b_{1\mu} = \frac{2}{L}\int_0^L sen\,(k\pi z/L)\,dz \frac{1}{\pi}\int_{-\pi}^{\pi}[w(\rho,\varphi,z) - u_e(\rho,\varphi,z)]\cos\mu\varphi\,d\varphi\,,$$

$$\sum_j J_\mu(\beta_{\mu j}\rho) b_{2\mu} = \frac{2}{L}\int_0^L sen\,(k\pi z/L)\,dz \frac{1}{\pi}\int_{-\pi}^{\pi}[w(\rho,\varphi,z) - u_e(\rho,\varphi,z)] sen\,\mu\varphi\,d\varphi\,.$$

Temos agora uma série em funções de Bessel, logo,

$$b_{10} = \frac{2}{a^2 J_1^2(\beta_{0j}a)}\int_0^a \rho J_0(\beta_{0j}\rho)d\rho \frac{2}{L}\int_0^L sen\,(k\pi z/L)\,dz\,\times$$
$$\times\frac{1}{2\pi}\int_{-\pi}^{\pi}[w(\rho,\varphi,z) - u_e(\rho,\varphi,z)]d\varphi\,,$$
$$b_{1\mu} = \frac{2}{a^2 J_{\mu+1}^2(\beta_{\mu j}a)}\int_0^a \rho J_\mu(\beta_{\mu j}\rho)d\rho \frac{2}{L}\int_0^L sen\,(k\pi z/L)\,dz\,\times$$
$$\times\frac{1}{\pi}\int_{-\pi}^{\pi}[w(\rho,\varphi,z) - u_e(\rho,\varphi,z)]\cos\mu\varphi\,d\varphi\,,$$
$$b_{2\mu} = \frac{2}{a^2 J_{\mu+1}^2(\beta_{\mu j}a)}\int_0^a \rho J_\mu(\beta_{\mu j}\rho)d\rho \frac{2}{L}\int_0^L sen\,(k\pi z/L)\,dz\,\times$$
$$\times\frac{1}{\pi}\int_{-\pi}^{\pi}[w(\rho,\varphi,z) - u_e(\rho,\varphi,z)] sen\,\mu\varphi\,d\varphi\,.$$

A solução é portanto,

$$
\begin{aligned}
u(t,\rho,\varphi,z) \;=\;& u_e(\rho,\varphi,z)+\sum_{jk}\exp\{-[(k\pi/L)^2+\beta_{0j}^2]\kappa t\}J_0(\beta_{0j}\rho)\times\\
&\times sen\,(k\pi z/L)\times\\
&\times\frac{2}{a^2J_1^2(\beta_{0j}a)}\int_0^a\rho J_0(\beta_{0j}\rho)d\rho\frac{2}{L}\int_0^L sen\,(k\pi z/L)\,dz\times\\
&\times\frac{1}{2\pi}\int_{-\pi}^{\pi}[w(\rho,\varphi,z)-u_e(\rho,\varphi,z)]d\varphi\\
&+\sum_{\mu jk}\exp\{-[(k\pi/L)^2+\beta_{\mu j}^2]\kappa t\}J_\mu(\beta_{\mu j}\rho)\times\\
&\times sen\,(k\pi z/L)\cos\mu\varphi\times\\
&\times\frac{2}{a^2J_{\mu+1}^2(\beta_{\mu j}a)}\int_0^a\rho J_\mu(\beta_{\mu j}\rho)d\rho\frac{2}{L}\int_0^L sen\,(k\pi z/L)\,dz\times\\
&\times\frac{1}{\pi}\int_{-\pi}^{\pi}[w(\rho,\varphi,z)-u_e(\rho,\varphi,z)]\cos\mu\varphi\,d\varphi\\
&+\sum_{\mu jk}\exp\{-[(k\pi/L)^2+\beta_{\mu j}^2]\kappa t\}J_\mu(\beta_{\mu j}\rho)\times\\
&\times sen\,(k\pi z/L)sen\,\mu\varphi\times\\
&\times\frac{2}{a^2J_{\mu+1}^2(\beta_{\mu j}a)}\int_0^a\rho J_\mu(\beta_{\mu j}\rho)d\rho\frac{2}{L}\int_0^L sen\,(k\pi z/L)\,dz\times\\
&\times\frac{1}{\pi}\int_{-\pi}^{\pi}[w(\rho,\varphi,z)-u_e(\rho,\varphi,z)]sen\,\mu\varphi\,d\varphi\,.
\end{aligned}
$$

Se $w(\rho,\varphi,z)=w_0$ *e* $u_e(\rho,\varphi,z)=u_e$ *constantes,*

$$
\begin{aligned}
u(t,\rho,\varphi,z) &= u_e + \sum_{jk} \exp\{-[(k\pi/L)^2 + \beta_{0j}^2]\kappa t\} J_0(\beta_{0j}\rho) \times \\
&\quad \times sen\,(k\pi z/L) \times \\
&\quad \times \frac{2}{a^2 J_1^2(\beta_{0j}a)} \int_0^a \rho J_0(\beta_{0j}\rho) d\rho \frac{2}{L} \int_0^L sen\,(k\pi z/L)\, dz (w_0 - u_e)\,, \\
&= u_e + \frac{4(w_0 - u_e)}{\pi} \sum_{jk} \exp\{-[((2k-1)\pi/L)^2 + \beta_{0j}^2]\kappa t\} J_0(\beta_{0j}\rho) \times \\
&\quad \times \frac{sen\,[(2k-1)\pi z/L]}{2k-1} \frac{2}{a^2 J_1^2(\beta_{0j}a)} \int_0^a \rho J_0(\beta_{0j}\rho) d\rho\,, \\
&= u_e + \frac{8(w_0 - u_e)}{a\pi} \sum_{jk} \exp\{-[((2k-1)\pi/L)^2 + \beta_{0j}^2]\kappa t\} \times \\
&\quad \times \frac{J_0(\beta_{0j}\rho)}{\beta_{0j} J_1(\beta_{0j}a)} \frac{sen\,[(2k-1)\pi z/L]}{2k-1}\,.
\end{aligned}
$$

Em $t = 0$,

$$
\begin{aligned}
u(t,\rho,\varphi,z) &= u_e + \frac{8(w_0 - u_e)}{a\pi} \sum_{jk} \frac{J_0(\beta_{0j}\rho)}{\beta_{0j} J_1(\beta_{0j}a)} \frac{sen\,[(2k-1)\pi z/L]}{2k-1}\,, \\
&= u_e + \frac{8(w_0 - u_e)}{a\pi} \sum_{j} \frac{J_0(\beta_{0j}\rho)}{\beta_{0j} J_1(\beta_{0j}a)} \sum_{k} \frac{sen\,[(2k-1)\pi z/L]}{2k-1}\,, \\
&= w_0\,,
\end{aligned}
$$

como esperado.

2. Consideremos um cilindro infinito de raio a. Calcule a temperatura $u(t,\rho,\varphi)$ sendo $u(t,a,\varphi) = \mu(t,\varphi)$, $u(0,\rho,\varphi) = \varphi(\rho,\varphi)$.

3. Consideremos a região $a \le \rho \le b$, $0 \le z \le L$. Calcule a temperatura $u(t,\rho,\varphi,z)$ sendo ([7], probl. 6.119 com $f = g = h = v = 0$),

$$
\begin{aligned}
&u(t,a,\varphi,z) = f(\varphi,z)\,, \quad u(t,b,\varphi,z) = g(\varphi,z)\,, \\
&u(t,\rho,\varphi,0) = h(\rho,\varphi)\,, \quad u(t,\rho,\varphi,L) = v(\rho,\varphi)\,, \\
&u(0,\rho,\varphi,z) = w(\rho,\varphi,z)\,.
\end{aligned}
$$

Escrevemos a solução como,

$$
\begin{aligned}
u(t,\rho,\varphi,z) &= u_e(\rho,\varphi,z) \\
&+\sum_{\beta\alpha\mu}\exp(-\lambda^2\kappa t)[c_1 J_\mu(\beta\rho)+c_2 Y_\mu(\beta\rho)]\times \\
&\times[d_1\cos\alpha z + d_2 sen\,\alpha z]\times \\
&\times[b_1\cos\mu\varphi + b_2 sen\,\mu\varphi]\,, \\
&\lambda^2-\alpha^2=\beta^2\,,\ \ \lambda^2>\alpha^2\,,
\end{aligned}
$$

e as condições de contorno,

$$
\begin{aligned}
u(t,a,\varphi,z) &= f(\varphi,z) = u_e(a,\varphi,z) \\
&+\sum_{\beta\alpha\mu}\exp(-\lambda^2\kappa t)[c_1 J_\mu(\beta a)+c_2 Y_\mu(\beta a)]\times \\
&\times[d_1\cos\alpha z + d_2 sen\,\alpha z]\times \\
&\times[b_1\cos\mu\varphi + b_2 sen\,\mu\varphi]\,, \\
u(t,b,\varphi,z) &= g(\varphi,z) = u_e(b,\varphi,z) \\
&+\sum_{\beta\alpha\mu}\exp(-\lambda^2\kappa t)[c_1 J_\mu(\beta b)+c_2 Y_\mu(\beta b)]\times \\
&\times[d_1\cos\alpha z + d_2 sen\,\alpha z]\times \\
&\times[b_1\cos\mu\varphi + b_2 sen\,\mu\varphi]\,, \\
u(t,\rho,\varphi,0) &= h(\rho,\varphi) = u_e(\rho,\varphi,0) \\
&+\sum_{\beta\alpha\mu}\exp(-\lambda^2\kappa t)[c_1 J_\mu(\beta\rho)+c_2 Y_\mu(\beta\rho)]d_1\times \\
&\times[b_1\cos\mu\varphi + b_2 sen\,\mu\varphi]\,, \\
u(t,\rho,\varphi,L) &= v(\rho,\varphi) = u_e(\rho,\varphi,L) \\
&+\sum_{\beta\alpha\mu}\exp(-\lambda^2\kappa t)[c_1 J_\mu(\beta\rho)+c_2 Y_\mu(\beta\rho)]\times \\
&\times[d_1\cos\alpha L + d_2 sen\,\alpha L]\times \\
&\times[b_1\cos\mu\varphi + b_2 sen\,\mu\varphi]\,, \\
u(0,\rho,\varphi,z) &= w(\rho,\varphi,z) = u_e(\rho,\varphi,z) \\
&+\sum_{\beta\alpha\mu}[c_1 J_\mu(\beta\rho)+c_2 Y_\mu(\beta\rho)]\times \\
&\times[d_1\cos\alpha z + d_2 sen\,\alpha z]\times \\
&\times[b_1\cos\mu\varphi + b_2 sen\,\mu\varphi]\,.
\end{aligned}
$$

Satisfazemos as condições acima escolhendo,

$$
\begin{aligned}
&c_1 J_\mu(\beta a) + c_2 Y_\mu(\beta a) = 0\,, \\
&c_1 J_\mu(\beta b) + c_2 Y_\mu(\beta b) = 0\,, \\
&d_1 = 0\,, \\
&\alpha_k L = k\pi\,, \quad k = 1, 2, \ldots
\end{aligned}
$$

Os valores de β *são determinados assim por,*

$$J_\mu(\beta_{\mu j} a) Y_\mu(\beta_{\mu j} b) - J_\mu(\beta_{\mu j} b) Y_\mu(\beta_{\mu j} a) = 0\,, \quad j = 1, 2, \ldots$$

ou,

$$U_\mu(\beta_{\mu j} b) = 0\,, \quad \mu = 0, 1, 2, \ldots, \quad j = 1, 2, \ldots$$

Também podemos escrever,

$$c_2 = -c_1 \frac{J_\mu(\beta a)}{Y_\mu(\beta a)}\,.$$

A solução fica então, fazendo $c_1 = d_2 = 1$*,*

$$
\begin{aligned}
u(t, \rho, \varphi, z) \;&=\; u_e(\rho, \varphi, z) + \sum_{\mu j k} \exp\{-[(k\pi/L)^2 + \beta_{\mu j}^2]\kappa t\} \frac{U_\mu(\beta_{\mu j}\rho)}{Y_\mu(\beta_{\mu j} a)} \times \\
&\times sen\,(k\pi z/L)[b_1 \cos \mu\varphi + b_2 sen\, \mu\varphi]\,.
\end{aligned}
$$

As condições de contorno ficam então,

$$
\begin{aligned}
f(\varphi, z) \;&=\; u_e(a, \varphi, z)\,, \\
g(\varphi, z) \;&=\; u_e(b, \varphi, z)\,, \\
h(\rho, \varphi) \;&=\; u_e(\rho, \varphi, 0)\,, \\
v(\rho, \varphi) \;&=\; u_e(\rho, \varphi, L)\,, \\
w(\rho, \varphi, z) \;&=\; u_e(\rho, \varphi, z) + \sum_{\mu j k} \frac{U_\mu(\beta_{\mu j}\rho)}{Y_\mu(\beta_{\mu j} a)} \times \\
&\times sen\,(k\pi z/L)[b_1 \cos \mu\varphi + b_2 sen\, \mu\varphi]\,.
\end{aligned}
$$

Temos séries de Fourier, séries de Fourier de senos e de Bessel, logo, considerando primeiro uma série de Fourier temos,

$$\sum_{jk} \frac{U_0(\beta_{0j}\rho)}{Y_0(\beta_{0j}a)} sen\,(k\pi z/L) b_{10} =$$
$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} [w(\rho,\varphi,z) - u_e(\rho,\varphi,z)]\, d\varphi\,,$$
$$\sum_{jk} \frac{U_\mu(\beta_{\mu j}\rho)}{Y_\mu(\beta_{\mu j}a)} sen\,(k\pi z/L) b_{1\mu} =$$
$$= \frac{1}{\pi} \int_{-\pi}^{\pi} [w(\rho,\varphi,z) - u_e(\rho,\varphi,z)] \cos\mu\varphi\, d\varphi\,,$$
$$\sum_{jk} \frac{U_\mu(\beta_{\mu j}\rho)}{Y_\mu(\beta_{\mu j}a)} sen\,(k\pi z/L) b_{2\mu} =$$
$$= \frac{1}{\pi} \int_{-\pi}^{\pi} [w(\rho,\varphi,z) - u_e(\rho,\varphi,z)] sen\,\mu\varphi\, d\varphi\,.$$

Considerando agora séries de Fourier de senos,

$$\sum_{j} \frac{U_0(\beta_{0j}\rho)}{Y_0(\beta_{0j}a)} b_{10} =$$
$$= \frac{2}{L} \int_0^L sen\,(k\pi z/L) dz \frac{1}{2\pi} \int_{-\pi}^{\pi} [w(\rho,\varphi,z) - u_e(\rho,\varphi,z)]\, d\varphi\,,$$
$$\sum_{j} \frac{U_\mu(\beta_{\mu j}\rho)}{Y_\mu(\beta_{\mu j}a)} b_{1\mu} =$$
$$= \frac{2}{L} \int_0^L sen\,(k\pi z/L) dz \frac{1}{\pi} \int_{-\pi}^{\pi} [w(\rho,\varphi,z) - u_e(\rho,\varphi,z)] \cos\mu\varphi\, d\varphi\,,$$
$$\sum_{j} \frac{U_\mu(\beta_{\mu j}\rho)}{Y_\mu(\beta_{\mu j}a)} b_{2\mu} =$$
$$= \frac{2}{L} \int_0^L sen\,(k\pi z/L) dz \frac{1}{\pi} \int_{-\pi}^{\pi} [w(\rho,\varphi,z) - u_e(\rho,\varphi,z)] sen\,\mu\varphi\, d\varphi\,.$$

Por fim, temos agora séries de funções de Bessel, logo,

$$b_{10} = \frac{Y_0(\beta_{0j}a)}{\displaystyle\int_a^b \rho U_0^2(\beta_{0j}\rho)d\rho} \int_a^b \rho U_0(\beta_{0j}\rho)d\rho \times$$
$$\times \frac{2}{L} \int_0^L sen\,(k\pi z/L)dz \frac{1}{2\pi} \int_{-\pi}^{\pi} [w(\rho,\varphi,z) - u_e(\rho,\varphi,z)]\, d\varphi\,,$$
$$b_{1\mu} = \frac{Y_\mu(\beta_{\mu j}a)}{\displaystyle\int_a^b \rho U_\mu^2(\beta_{\mu j}\rho)d\rho} \int_a^b \rho U_\mu(\beta_{\mu j}\rho)d\rho \times$$
$$\times \frac{2}{L} \int_0^L sen\,(k\pi z/L)dz \frac{1}{\pi} \int_{-\pi}^{\pi} [w(\rho,\varphi,z) - u_e(\rho,\varphi,z)] \cos\mu\varphi\, d\varphi\,,$$
$$b_{2\mu} = \frac{Y_\mu(\beta_{\mu j}a)}{\displaystyle\int_a^b \rho U_\mu^2(\beta_{\mu j}\rho)d\rho} \int_a^b \rho U_\mu(\beta_{\mu j}\rho)d\rho \times$$
$$\times \frac{2}{L} \int_0^L sen\,(k\pi z/L)dz \frac{1}{\pi} \int_{-\pi}^{\pi} [w(\rho,\varphi,z) - u_e(\rho,\varphi,z)] sen\,\mu\varphi\, d\varphi\,.$$

A solução é então,

$$
\begin{aligned}
u(t,\rho,\varphi,z) &= u_e(\rho,\varphi,z) + \sum_{jk} \exp\{-[(k\pi/L)^2 + \beta_{0j}^2]\kappa t\} \times \\
&\times sen\,(k\pi z/L) \frac{U_0(\beta_{0j}\rho)}{\displaystyle\int_a^b \rho U_0^2(\beta_{0j}\rho) d\rho} \int_a^b \rho U_0(\beta_{0j}\rho) d\rho \times \\
&\times \frac{2}{L} \int_0^L sen\,(k\pi z/L) dz \frac{1}{2\pi} \int_{-\pi}^{\pi} [w(\rho,\varphi,z) - u_e(\rho,\varphi,z)]\, d\varphi \\
&+ \sum_{\mu jk} \exp\{-[(k\pi/L)^2 + \beta_{\mu j}^2]\kappa t\} sen\,(k\pi z/L) \cos\mu\varphi \times \\
&\times \frac{U_\mu(\beta_{\mu j}\rho)}{\displaystyle\int_a^b \rho U_\mu^2(\beta_{\mu j}\rho) d\rho} \int_a^b \rho U_\mu(\beta_{\mu j}\rho) d\rho \times \\
&\times \frac{2}{L} \int_0^L sen\,(k\pi z/L) dz \frac{1}{\pi} \int_{-\pi}^{\pi} [w(\rho,\varphi,z) - u_e(\rho,\varphi,z)] \cos\mu\varphi\, d\varphi \\
&+ \sum_{\mu jk} \exp\{-[(k\pi/L)^2 + \beta_{\mu j}^2]\kappa t\} sen\,(k\pi z/L) sen\,\mu\varphi \times \\
&\times \frac{U_\mu(\beta_{\mu j}\rho)}{\displaystyle\int_a^b \rho U_\mu^2(\beta_{\mu j}\rho) d\rho} \int_a^b \rho U_\mu(\beta_{\mu j}\rho) d\rho \times \\
&\times \frac{2}{L} \int_0^L sen\,(k\pi z/L) dz \frac{1}{\pi} \int_{-\pi}^{\pi} [w(\rho,\varphi,z) - u_e(\rho,\varphi,z)] sen\,\mu\varphi\, d\varphi\,.
\end{aligned}
$$

Se $w = w_0$ e u_e são constantes,

$$
\begin{aligned}
u(t,\rho,\varphi,z) &= u_e + \sum_{jk} \exp\{-[(k\pi/L)^2 + \beta_{0j}^2]\kappa t\} \times \\
&\quad \times sen\,(k\pi z/L) \frac{U_0(\beta_{0j}\rho)}{\displaystyle\int_a^b \rho U_0^2(\beta_{0j}\rho) d\rho} \int_a^b \rho U_0(\beta_{0j}\rho) d\rho \times \\
&\quad \times \frac{2}{L} \int_0^L sen\,(k\pi z/L) dz (w_0 - u_e)\,, \\
&= u_e + \frac{4(w_0 - u_e)}{\pi} \sum_{jk} \exp\{-[((2k-1)\pi/L)^2 + \beta_{0j}^2]\kappa t\} \times \\
&\quad \times \frac{sen\,[(2k-1)\pi z/L]}{2k-1} \frac{U_0(\beta_{0j}\rho)}{\displaystyle\int_a^b \rho U_0^2(\beta_{0j}\rho) d\rho} \int_a^b \rho U_0(\beta_{0j}\rho) d\rho\,.
\end{aligned}
$$

Em $t = 0$*,*

$$
\begin{aligned}
u(t,\rho,\varphi,z) &= u_e + \frac{4(w_0 - u_e)}{\pi} \sum_{k} \frac{sen\,[(2k-1)\pi z/L]}{2k-1} \times \\
&\quad \times \sum_{j} \frac{U_0(\beta_{0j}\rho)}{\displaystyle\int_a^b \rho U_0^2(\beta_{0j}\rho) d\rho} \int_a^b \rho U_0(\beta_{0j}\rho) d\rho = w_0\,,
\end{aligned}
$$

como esperado.

4. Calcule $u(t,\rho,\varphi,z)$ para,

(a)

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + F(\rho,\varphi,z)\,,$$

(b)

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + \alpha^2\,,$$

com α constante,

(c)

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + \alpha^2 u\,,$$

com α constante, e,

$$
\begin{aligned}
&0 \le \rho \le a\,, \quad 0 \le \varphi \le 2\pi\,, \quad 0 \le z \le L\,,\\
&u(t,a,\varphi,z) = \mu_1(\varphi,z)\,,\\
&u(t,\rho,\varphi,0) = \nu_1(\rho,\varphi)\,, \quad u(t,\rho,\varphi,L) = \nu_2(\rho,\varphi)\,,\\
&u(0,\rho,\varphi,z) = f(\rho,\varphi,z)\,.
\end{aligned}
$$

5. Calcule $u(t,\rho,\varphi,z)$ para,

(a)

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + F(t,\rho,\varphi,z)\,,$$

(b)

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u\,,$$

com α constante,

(c)

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + \alpha^2\,,$$

com α constante,

(d)

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + \alpha^2 u\,,$$

e,

$$
\begin{aligned}
&0 \le \rho \le a\,, \quad 0 \le \varphi \le 2\pi\,, \quad 0 \le z \le L\,,\\
&u(t,a,\varphi,z) = \mu_1(t,\varphi,z)\,,\\
&u(t,\rho,\varphi,0) = \nu_1(t,\rho,\varphi)\,, \quad u(t,\rho,\varphi,L) = \nu_2(t,\rho,\varphi)\,,\\
&u(0,\rho,\varphi,z) = f(\rho,\varphi,z)\,.
\end{aligned}
$$

6. Calcule $u(t,\rho,\varphi,z)$ para,

(a)

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + F(\rho,\varphi,z)\,,$$

(b)

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + \alpha^2\,,$$

com α constante,

(c)

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + \alpha^2 u\,,$$

com α constante, e,

$$
\begin{aligned}
& a \leq \rho \leq b\,, \quad 0 \leq \varphi \leq 2\pi\,, \quad 0 \leq z \leq L\,, \\
& u(t,a,\varphi,z) = \mu_1(\varphi,z)\,, \quad u(t,b,\varphi,z) = \mu_2(\varphi,z)\,, \\
& u(t,\rho,\varphi,0) = \nu_1(\rho,\varphi)\,, \quad u(t,\rho,\varphi,L) = \nu_2(\rho,\varphi)\,, \\
& u(0,\rho,\varphi,z) = f(\rho,\varphi,z)\,.
\end{aligned}
$$

7. Calcule $u(t,\rho,\varphi,z)$ para,

(a)

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + F(t,\rho,\varphi,z)\,,$$

(b)

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u\,,$$

(c)

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + \alpha^2\,,$$

com α constante,

(d)

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + \alpha^2 u\,,$$

com α constante, e,

$$
\begin{aligned}
& a \leq \rho \leq b\,, \quad 0 \leq \varphi \leq 2\pi\,, \quad 0 \leq z \leq L\,, \\
& u(t,a,\varphi,z) = \mu_1(t,\varphi,z)\,, \quad u(t,b,\varphi,z) = \mu_2(t,\varphi,z)\,, \\
& u(t,\rho,\varphi,0) = \nu_1(t,\rho,\varphi)\,, \quad u(t,\rho,\varphi,L) = \nu_2(t,\rho,\varphi)\,, \\
& u(0,\rho,\varphi,z) = f(\rho,\varphi,z)\,.
\end{aligned}
$$

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