4 - A equação de Laplace em coordenadas cilíndricas

A equação de Laplace, $\nabla^2 u = 0$, em coordenadas cilíndricas, é,

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2} + \frac{\partial^2 u}{\partial z^2} = 0\,, \tag{1}$$

com $u = u(\rho, \varphi, z)$. Procederemos de maneira análoga ao capítulo 3, procurando soluções que descrevam determinados problemas físicos.

1 Considerando $u = u(\rho)$

A equação (1) fica, considerando $u = u(\rho)$,

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{du}{d\rho}\right) = 0\,.$$

Resolvendo a equação acima temos,

$$\rho\frac{du}{d\rho} = c_0\,,$$

com c_0 uma constante. Continuando,

$$\frac{du}{d\rho} = \frac{c_0}{\rho}\,,$$
$$u = c_0 \ln\rho + c_1\,,$$

com c_1 outra constante de integração. A solução da equação de Laplace em coordenadas cilíndricas, considerando $u = u(\rho)$ apenas, é então,

$$u(\rho) = c_0 \ln\rho + c_1\,. \tag{2}$$

Intervalo $0 \leq \rho \leq \infty$

A solução (2) diverge em $\rho \to 0$ e $\rho \to \infty$. Embora isso seja inconveniente para um sistema físico real, em algumas situações idealizadas essa função é útil. O potencial eletrostático para uma linha carregada infinita, por exemplo, é da forma (2). Nesse caso é comum definirmos um ponto ρ_0 como o ponto em que o potencial se anula. Temos,

$$u(\rho_0) = c_0 \ln \rho_0 + c_1 = 0\,,$$
$$c_1 = -c_0 \ln \rho_0\,,$$

e portanto,

$$u(\rho) = c_0 \ln \rho - c_0 \ln \rho_0 = c_0 \ln(\rho/\rho_0)\,.$$

Resta ainda o problema de determinar c_0.

Intervalo $a \leq \rho \leq b$

Agora não temos problemas de divergências.

Condição de contorno $u(a) = u_a$, $u(b) = u_b$, com $a < b$

Com as condições acima obtemos o sistema,

$$u(a) = c_0 \ln a + c_1 = u_a\,,$$
$$u(b) = c_0 \ln b + c_1 = u_b\,,$$

com soluccão,

$$c_0 = \frac{u_b - u_a}{\ln(b/a)}\,,$$
$$c_1 = \frac{u_a \ln b - u_b \ln a}{\ln(b/a)}\,.$$

se $u_a = u_b$ temos $c_0 = 0$, $c_1 = u_a$ e $u(\rho) = u_a$ constante. Condições de contorno envolvendo a derivada de u podem ser tratadas de forma semelhante.

2 Problemas

1. Encontre a solução $u(\rho)$ para as equações:

   (a)
$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{du}{d\rho}\right) = -f(\rho)\,;$$

(b)
$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{du}{d\rho}\right) = +\alpha^2\,;$$
com α constante,

(c)
$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{du}{d\rho}\right) = -\alpha^2\,;$$
com α constante,

(d)
$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{du}{d\rho}\right) = +\alpha^2 u\,;$$
com α constante,

(e)
$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{du}{d\rho}\right) = -\alpha^2 u\,;$$
com α constante e,

$$0 \le \rho \le a\,,$$
$$u(a) = u_a\,.$$

(a) A equação fica,

$$\frac{d^2u}{d\rho^2} + \frac{1}{\rho}\frac{du}{d\rho} = -f(\rho)\,.$$

Escrevemos a solução como,

$$u(\rho) = u_h(\rho) + u_p(\rho)\,,$$

em que u_h é a solução da equação homogênea com condições de contorno não-homogêneas, e u_p é uma solução particular da equação não-homogênea com condições de contorno homogêneas. Vimos que a solução da equação homogênea em $0 \le \rho \le a$, finita, é uma constante, logo $u_h(\rho) = u_a =$ constante. Expandimos u_p e f em séries de funções de Bessel,

$$u_p(\rho) = \sum_{i=1}^{\infty} A_i J_n(\lambda_i \rho)\,, \quad 0 \le \rho \le a\,,$$
$$A_i = \frac{2}{a^2 J_{n+1}^2(\lambda_i a)} \int_0^a \rho' J_n(\lambda_i \rho') u_p(\rho') d\rho'\,.$$

$$f(\rho) = \sum_{i=1}^{\infty} B_i J_n(\lambda_i \rho)\,, \quad 0 \leq \rho \leq a\,,$$

$$B_i = \frac{2}{a^2 J_{n+1}^2(\lambda_i a)} \int_0^a \rho' J_n(\lambda_i \rho') f(\rho') d\rho'\,.$$

$$J_n(\lambda_i a) = 0\,, \quad i = 1, 2, 3, \ldots\,.$$

Substituindo u_p *e* f *na equação diferencial,*

$$\begin{aligned}
&\frac{d^2 u_p}{d\rho^2} + \frac{1}{\rho}\frac{du_p}{d\rho} = -f(\rho)\,,\\
&\frac{d^2}{d\rho^2}\left[\sum_{i=1}^{\infty} A_i J_n(\lambda_i \rho)\right] + \frac{1}{\rho}\frac{d}{d\rho}\left[\sum_{i=1}^{\infty} A_i J_n(\lambda_i \rho)\right] = -\sum_{i=1}^{\infty} B_i J_n(\lambda_i \rho)\,,\\
&\sum_{i=1}^{\infty} A_i \frac{d^2}{d\rho^2}\left[J_n(\lambda_i \rho)\right] + \frac{1}{\rho}\sum_{i=1}^{\infty} A_i \frac{d}{d\rho}\left[J_n(\lambda_i \rho)\right] = -\sum_{i=1}^{\infty} B_i J_n(\lambda_i \rho)\,.
\end{aligned}$$

Usando as relações [6, 7, 8],

$$x J_n'(x) = n J_n(x) - x J_{n+1}(x)\,,$$

ou,

$$J_n'(x) = \frac{n}{x} J_n(x) - J_{n+1}(x)\,,$$

temos,

$$\begin{aligned}
\frac{d}{d\rho}\left[J_n(\lambda_i \rho)\right] &= \lambda_i \left[\frac{n}{\lambda_i \rho} J_n(\lambda_i \rho) - J_{n+1}(\lambda_i \rho)\right]\,,\\
&= \frac{n}{\rho} J_n(\lambda_i \rho) - \lambda_i J_{n+1}(\lambda_i \rho)\,,
\end{aligned}$$

$$
\begin{aligned}
\frac{d^2}{d\rho^2}\left[J_n(\lambda_i\rho)\right] &= \lambda_i^2\left[-\frac{n}{(\lambda_i\rho)^2}J_n(\lambda_i\rho)+\frac{n}{\lambda_i\rho}\left(\frac{n}{\lambda_i\rho}J_n(\lambda_i\rho)-J_{n+1}(\lambda_i\rho)\right)\right.\\
&\left.-\frac{n+1}{\lambda_i\rho}J_{n+1}(\lambda_i\rho)+J_{n+2}(\lambda_i\rho)\right],\\
&= -\frac{n}{\rho^2}J_n(\lambda_i\rho)+\frac{n^2}{\rho^2}J_n(\lambda_i\rho)-\frac{n\lambda_i}{\rho}J_{n+1}(\lambda_i\rho)\\
&-\frac{(n+1)\lambda_i}{\rho}J_{n+1}(\lambda_i\rho)+\lambda_i^2J_{n+2}(\lambda_i\rho),\\
&= \frac{n(n-1)}{\rho^2}J_n(\lambda_i\rho)-\frac{(2n+1)\lambda_i}{\rho}J_{n+1}(\lambda_i\rho)\\
&+\lambda_i^2J_{n+2}(\lambda_i\rho).
\end{aligned}
$$

A equação diferencial fica então,

$$
\begin{aligned}
&\sum_{i=1}^{\infty}A_i\frac{d^2}{d\rho^2}\left[J_n(\lambda_i\rho)\right]+\frac{1}{\rho}\sum_{i=1}^{\infty}A_i\frac{d}{d\rho}\left[J_n(\lambda_i\rho)\right]=-\sum_{i=1}^{\infty}B_iJ_n(\lambda_i\rho),\\
&\sum_{i=1}^{\infty}A_i\left[\frac{n(n-1)}{\rho^2}J_n(\lambda_i\rho)-\frac{(2n+1)\lambda_i}{\rho}J_{n+1}(\lambda_i\rho)+\lambda_i^2J_{n+2}(\lambda_i\rho)\right]\\
&+\frac{1}{\rho}\sum_{i=1}^{\infty}A_i\left[\frac{n}{\rho}J_n(\lambda_i\rho)-\lambda_iJ_{n+1}(\lambda_i\rho)\right]=-\sum_{i=1}^{\infty}B_iJ_n(\lambda_i\rho),\\
&\sum_{i=1}^{\infty}A_i\left[\frac{n^2}{\rho^2}J_n(\lambda_i\rho)-\frac{(2n+2)\lambda_i}{\rho}J_{n+1}(\lambda_i\rho)+\lambda_i^2J_{n+2}(\lambda_i\rho)\right]=\\
&=-\sum_{i=1}^{\infty}B_iJ_n(\lambda_i\rho).
\end{aligned}
$$

Usando,

$$
J_{n+1}(x)=\frac{2n}{x}J_n(x)-J_{n-1}(x),
$$

temos,

$$
J_{n+2}(x)=\frac{2(n+1)}{x}J_{n+1}(x)-J_n(x),
$$

logo,

$$\sum_{i=1}^{\infty} A_i \left[\frac{n^2}{\rho^2} J_n(\lambda_i \rho) - \frac{(2n+2)\lambda_i}{\rho} J_{n+1}(\lambda_i \rho) + \lambda_i^2 \left(\frac{2(n+1)}{\lambda_i \rho} J_{n+1}(\lambda_i \rho) - J_n(\lambda_i \rho) \right) \right] = = -\sum_{i=1}^{\infty} B_i J_n(\lambda_i \rho)\,,$$

$$\sum_{i=1}^{\infty} A_i \left[\frac{n^2}{\rho^2} J_n(\lambda_i \rho) - \lambda_i^2 J_n(\lambda_i \rho) \right] = = -\sum_{i=1}^{\infty} B_i J_n(\lambda_i \rho)\,.$$

Portanto,

$$A_i \left[\frac{n^2}{\rho^2} - \lambda_i^2 \right] = -B_i\,. \tag{3}$$

Poderíamos ter escrito (3) desde o início, pois J_n é solução da equação de Bessel. Supomos que A_i é independente de ρ, logo devemos ter $n = 0$,

$$A_i = \frac{B_i}{\lambda_i^2} = \frac{1}{\lambda_i^2} \frac{2}{a^2 J_1^2(\lambda_i a)} \int_0^a \rho' J_0(\lambda_i \rho') f(\rho') d\rho'\,.$$

A solução particular u_p é então,

$$\begin{aligned} u_p(\rho) &= \sum_{i=1}^{\infty} A_i J_0(\lambda_i \rho) = \sum_{i=1}^{\infty} \frac{B_i}{\lambda_i^2} J_0(\lambda_i \rho)\,, \\ &= \sum_{i=1}^{\infty} J_0(\lambda_i \rho) \frac{1}{\lambda_i^2} \frac{2}{a^2 J_1^2(\lambda_i a)} \int_0^a \rho' J_0(\lambda_i \rho') f(\rho') d\rho'\,, \\ & \quad 0 \le \rho \le a\,. \end{aligned}$$

Notemos que a solução u_p satisfaz condições de contorno homogêneas em $\rho = a$, como esperado.

Podemos escrever u_p em termos de uma função de Green,

$$u_p(\rho) = \int_0^a G(\rho, \rho') f(\rho') d\rho'\,,$$

com,

$$
\begin{aligned}
G(\rho,\rho') &= \sum_{i=1}^{\infty} J_0(\lambda_i\rho)\frac{1}{\lambda_i^2}\frac{2}{a^2J_1^2(\lambda_i a)}\rho' J_0(\lambda_i\rho')\,, \\
&= \frac{2}{a^2}\sum_{i=1}^{\infty}\frac{\rho'}{\lambda_i^2}\frac{J_0(\lambda_i\rho)J_0(\lambda_i\rho')}{J_1^2(\lambda_i a)}\,, \\
& \quad 0\le\rho\le a\,.
\end{aligned}
$$

Notemos que, se $f(\rho)=f_0$ constante,

$$
\begin{aligned}
u_p(\rho) &= \sum_{i=1}^{\infty} J_0(\lambda_i\rho)\frac{1}{\lambda_i^2}\frac{2}{a^2J_1^2(\lambda_i a)}\int_0^a \rho' J_0(\lambda_i\rho')f(\rho')d\rho'\,, \\
&= f_0\sum_{i=1}^{\infty} J_0(\lambda_i\rho)\frac{1}{\lambda_i^2}\frac{2}{a^2J_1^2(\lambda_i a)}\int_0^a \rho' J_0(\lambda_i\rho')d\rho'\,, \\
& \quad 0\le\rho\le a\,.
\end{aligned}
$$

Usando as relações,

$$
\int xJ_0(x)dx = xJ_1(x)\,,
$$

$$
\int x^nJ_{n-1}(x)dx = x^nJ_n(x)\,,
$$

temos,

$$
\int_0^a \rho' J_0(\lambda_i\rho')d\rho' = \frac{1}{\lambda_i^2}\int_0^{\lambda_i a} uJ_0(u)du = \frac{a}{\lambda_i}J_1(\lambda_i a)\,.
$$

Portanto,

$$
\begin{aligned}
u_p(\rho) &= f_0\sum_{i=1}^{\infty} J_0(\lambda_i\rho)\frac{1}{\lambda_i^2}\frac{2}{a^2J_1^2(\lambda_i a)}\int_0^a \rho' J_0(\lambda_i\rho')d\rho'\,, \\
&= f_0\sum_{i=1}^{\infty} J_0(\lambda_i\rho)\frac{1}{\lambda_i^2}\frac{2}{a^2J_1^2(\lambda_i a)}\frac{a}{\lambda_i}J_1(\lambda_i a)\,, \\
&= f_0\frac{2}{a}\sum_{i=1}^{\infty}\frac{1}{\lambda_i^3}\frac{J_0(\lambda_i\rho)}{J_1(\lambda_i a)}\,, \\
& \quad 0\le\rho\le a\,.
\end{aligned}
$$

Podemos verificar que a solução acima satisfaz a equação diferencial,

$$
\begin{aligned}
&\frac{d^2u_p}{d\rho^2} + \frac{1}{\rho}\frac{du_p}{d\rho} = -f(\rho)\,, \\
&\frac{d^2}{d\rho^2}\left[f_0\frac{2}{a}\sum_{i=1}^{\infty}\frac{1}{\lambda_i^3}\frac{J_0(\lambda_i\rho)}{J_1(\lambda_i a)}\right] + \frac{1}{\rho}\frac{d}{d\rho}\left[f_0\frac{2}{a}\sum_{i=1}^{\infty}\frac{1}{\lambda_i^3}\frac{J_0(\lambda_i\rho)}{J_1(\lambda_i a)}\right] = -f_0\,, \\
&f_0\frac{2}{a}\sum_{i=1}^{\infty}\frac{1}{\lambda_i^3}\frac{1}{J_1(\lambda_i a)}\frac{d^2}{d\rho^2}J_0(\lambda_i\rho) + \frac{1}{\rho}f_0\frac{2}{a}\sum_{i=1}^{\infty}\frac{1}{\lambda_i^3}\frac{1}{J_1(\lambda_i a)}\frac{d}{d\rho}J_0(\lambda_i\rho) = -f_0\,, \\
&f_0\frac{2}{a}\sum_{i=1}^{\infty}\frac{1}{\lambda_i^3}\frac{1}{J_1(\lambda_i a)}\left[\frac{d^2}{d\rho^2}J_0(\lambda_i\rho) + \frac{1}{\rho}\frac{d}{d\rho}J_0(\lambda_i\rho)\right] = -f_0\,.
\end{aligned}
$$

Como $J_0(\lambda_i\rho)$ é solução da equação de Bessel de ordem zero,

$$
\frac{d^2}{d\rho^2}J_0(\lambda_i\rho) + \frac{1}{\rho}\frac{d}{d\rho}J_0(\lambda_i\rho) = -\lambda_i^2 J_0(\lambda_i\rho)\,,
$$

logo,

$$
\begin{aligned}
&f_0\frac{2}{a}\sum_{i=1}^{\infty}\frac{1}{\lambda_i^3}\frac{1}{J_1(\lambda_i a)}\left[-\lambda_i^2 J_0(\lambda_i\rho)\right] = -f_0\,, \\
&\frac{2}{a}\sum_{i=1}^{\infty}\frac{1}{\lambda_i}\frac{J_0(\lambda_i\rho)}{J_1(\lambda_i a)} = 1\,,
\end{aligned}
$$

como deve ser.

Usando a relação,

$$
\sum_{i=1}^{\infty}\frac{1}{\lambda_i^3}\frac{J_0(\lambda_i\rho)}{J_1(\lambda_i a)} = \frac{a}{8}(a^2-\rho^2)\,,
$$

podemos escrever u_p para f constante como,

$$
\begin{aligned}
u_p(\rho) &= f_0\frac{2}{a}\sum_{i=1}^{\infty}\frac{1}{\lambda_i^3}\frac{J_0(\lambda_i\rho)}{J_1(\lambda_i a)}\,, \\
&= f_0\frac{2}{a}\frac{a}{8}(a^2-\rho^2) = f_0\frac{1}{4}(a^2-\rho^2)\,, \\
&\ 0 \le \rho \le a\,.
\end{aligned}
$$

A solução acima satisfaz de fato a equação diferencial.

(b) Usando o resultado anterior com $-f(\rho) = +\alpha^2$ constante, a solução particular é,

$$
\begin{aligned}
u_p(\rho) &= -\alpha^2 \frac{1}{4}(a^2 - \rho^2)\,, \\
& \quad 0 \leq \rho \leq a\,.
\end{aligned}
$$

(c) Nesse caso temos,

$$
\begin{aligned}
u_p(\rho) &= +\alpha^2 \frac{1}{4}(a^2 - \rho^2)\,, \\
& \quad 0 \leq \rho \leq a\,.
\end{aligned}
$$

(d) Temos,

$$
\begin{aligned}
&\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{du}{d\rho}\right) = +\alpha^2 u\,, \\
&\frac{d^2u}{d\rho^2} + \frac{1}{\rho}\frac{du}{d\rho} = +\alpha^2 u\,, \\
&\rho^2\frac{d^2u}{d\rho^2} + \rho\frac{du}{d\rho} - \alpha^2\rho^2 u = 0\,.
\end{aligned}
$$

Temos uma equação de Bessel modificada de ordem zero, com solução,

$$u(\rho) = c_1 I_0(\alpha\rho) + c_2 K_0(\alpha\rho)\,,$$

em que I_0 e K_0 são as funções de Bessel modificadas de ordem zero, de primeira e segunda espécie, respectivamente. A solução finita em $0 \leq \rho a$ é obtida fazendo $c_2 = 0$,

$$u(\rho) = c_1 I_0(\alpha\rho)\,.$$

Em $\rho = a$,

$$u(a) = c_1 I_0(\alpha a) = u_a\,,$$

logo,

$$c_1 = \frac{u_a}{I_0(\alpha a)}\,.$$

A solução é portanto,

$$u(\rho) = u_a \frac{I_0(\alpha\rho)}{I_0(\alpha a)}\,.$$

(e) Agora temos a equação de Bessel de ordem zero,

$$\rho^2 \frac{d^2u}{d\rho^2} + \rho \frac{du}{d\rho} + \alpha^2 \rho^2 u = 0\,,$$

com solução,

$$u(\rho) = c_1 J_0(\alpha\rho) + c_2 Y_0(\alpha\rho)\,,$$

em que J_0 e Y_0 são as funções de Bessel de ordem zero, de primeira e segunda espécie, respectivamente. A solução finita em $0 \leq \rho a$ é obtida fazendo $c_2 = 0$,

$$u(\rho) = c_1 J_0(\alpha\rho)\,.$$

Em $\rho = a$,

$$u(a) = c_1 J_0(\alpha a) = u_a\,,$$

logo,

$$c_1 = \frac{u_a}{J_0(\alpha a)}\,.$$

A solução é portanto,

$$u(\rho) = u_a \frac{J_0(\alpha\rho)}{J_0(\alpha a)}\,.$$

2. Encontre a solução $u(\rho)$ para as equações:

(a)

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho \frac{du}{d\rho}\right) = -f(\rho)\,;$$

(b)

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho \frac{du}{d\rho}\right) = +\alpha^2\,;$$

com α constante,

(c)

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho \frac{du}{d\rho}\right) = -\alpha^2\,;$$

com α constante,

(d)

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{du}{d\rho}\right) = +\alpha^2 u\,;$$

com α constante,

(e)

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{du}{d\rho}\right) = -\alpha^2 u\,;$$

com α constante e,

$$\begin{aligned} & a \leq \rho \leq b\,, \\ & u(a) = u_a\,, \quad u(b) = u_b\,. \end{aligned}$$

(a) A equação fica,

$$\frac{d^2u}{d\rho^2} + \frac{1}{\rho}\frac{du}{d\rho} = -f(\rho)\,.$$

Escrevemos a solução, como antes,

$$u(\rho) = u_h(\rho) + u_p(\rho)\,,$$

em que u_h é a solução da equação homogênea com condições de contorno não-homogêneas, e u_p é uma solução particular da equação não-homogênea com condições de contorno homogêneas. Expandimos u_p e f em séries de funções de Bessel,

$$U_n(\lambda_j\rho) \equiv J_n(\lambda_j\rho)Y_n(\lambda_j a) - J_n(\lambda_j a)Y_n(\lambda_j\rho)\,,$$

com λ_j definido por,

$$U_n(\lambda_j b) = 0\,,$$

As funções U_n são ortogonais,

$$\int_a^b \rho U_n(\lambda_j\rho)U_n(\lambda_k\rho)d\rho = 0\,, \quad j \neq k\,.$$

As expansões para u_p e f são,

$$u_p(\rho) = \sum_{j=1} A_j U_n(\lambda_j\rho)\,,$$

$$A_j = \frac{\displaystyle\int_a^b \rho' u_p(\rho') U_n(\lambda_j \rho') d\rho'}{\displaystyle\int_a^b \rho' U_n^2(\lambda_j \rho') d\rho'}\,,$$

$$f(\rho) = \sum_{j=1} B_j U_n(\lambda_j \rho)\,,$$

$$B_j = \frac{\displaystyle\int_a^b \rho' f(\rho') U_n(\lambda_j \rho') d\rho'}{\displaystyle\int_a^b \rho' U_n^2(\lambda_j \rho') d\rho'}\,.$$

Substituindo u_p e f na equação diferencial,

$$\begin{aligned}
&\frac{d^2 u_p}{d\rho^2} + \frac{1}{\rho}\frac{du_p}{d\rho} = -f(\rho)\,,\\
&\frac{d^2}{d\rho^2}\left[\sum_{j=1} A_j U_n(\lambda_j\rho)\right] + \frac{1}{\rho}\frac{d}{d\rho}\left[\sum_{j=1} A_j U_n(\lambda_j\rho)\right] = -\sum_{j=1} B_j U_n(\lambda_j\rho)\,,\\
&\sum_{j=1} A_j \frac{d^2}{d\rho^2} U_n(\lambda_j\rho) + \sum_{j=1} A_j \frac{1}{\rho}\frac{d}{d\rho} U_n(\lambda_j\rho) = -\sum_{j=1} B_j U_n(\lambda_j\rho)\,,\\
&\sum_{j=1} A_j \left[\frac{d^2}{d\rho^2} U_n(\lambda_j\rho) + \frac{1}{\rho}\frac{d}{d\rho} U_n(\lambda_j\rho)\right] = -\sum_{j=1} B_j U_n(\lambda_j\rho)\,,\\
&\sum_{j=1} A_j \left[(n^2/\rho^2) - \lambda_j^2\right] U_n(\lambda_j\rho) = -\sum_{j=1} B_j U_n(\lambda_j\rho)\,.
\end{aligned}$$

pois U_n satisfaz a equação de Bessel. Portanto,

$$A_j \left[(n^2/\rho^2) - \lambda_j^2\right] = -B_j\,.$$

Como antes, devemos ter $n = 0$ para A_j constante, logo,

$$A_j = \frac{B_j}{\lambda_j^2} = \frac{1}{\lambda_j^2}\frac{\displaystyle\int_a^b \rho' f(\rho') U_0(\lambda_j \rho') d\rho'}{\displaystyle\int_a^b \rho' U_0^2(\lambda_j \rho') d\rho'}\,.$$

A solução u_p é portanto,

$$
\begin{aligned}
u_p(\rho) &= \sum_{j=1} A_j U_0(\lambda_j \rho) = \sum_{j=1} \frac{B_j}{\lambda_j^2} U_0(\lambda_j \rho)\,, \\
&= \sum_{j=1} \frac{U_0(\lambda_j \rho)}{\lambda_j^2} \frac{\displaystyle\int_a^b \rho' f(\rho') U_0(\lambda_j \rho') d\rho'}{\displaystyle\int_a^b \rho' U_0^2(\lambda_j \rho') d\rho'}\,.
\end{aligned}
$$

Escrevendo u_p em termos de função de Green temos,

$$
u_p(\rho) = \int_a^b G(\rho, \rho') f(\rho') d\rho'\,,
$$

com,

$$
G(\rho, \rho') = \sum_{j=1} \frac{1}{\lambda_j^2} \frac{\rho' U_0(\lambda_j \rho) U_0(\lambda_j \rho')}{\displaystyle\int_a^b \rho' U_0^2(\lambda_j \rho') d\rho'}\,.
$$

Se $f(\rho) = f_0$ constante temos,

$$
u_p(\rho) = f_0 \sum_{j=1} \frac{U_0(\lambda_j \rho)}{\lambda_j^2} \frac{\displaystyle\int_a^b \rho' U_0(\lambda_j \rho') d\rho'}{\displaystyle\int_a^b \rho' U_0^2(\lambda_j \rho') d\rho'}\,,
$$

(b) Com $f(\rho) = -\alpha^2$,

$$
u_p(\rho) = -\alpha^2 \sum_{j=1} \frac{U_0(\lambda_j \rho)}{\lambda_j^2} \frac{\displaystyle\int_a^b \rho' U_0(\lambda_j \rho') d\rho'}{\displaystyle\int_a^b \rho' U_0^2(\lambda_j \rho') d\rho'}\,.
$$

(c) Agora $f(\rho) = +\alpha^2$, logo,

$$
u_p(\rho) = +\alpha^2 \sum_{j=1} \frac{U_0(\lambda_j \rho)}{\lambda_j^2} \frac{\displaystyle\int_a^b \rho' U_0(\lambda_j \rho') d\rho'}{\displaystyle\int_a^b \rho' U_0^2(\lambda_j \rho') d\rho'}\,.
$$

(d) Temos,

$$
\begin{aligned}
&\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{du}{d\rho}\right) = +\alpha^2 u\,, \\
&\frac{d^2u}{d\rho^2} + \frac{1}{\rho}\frac{du}{d\rho} = +\alpha^2 u\,, \\
&\rho^2\frac{d^2u}{d\rho^2} + \rho\frac{du}{d\rho} - \alpha^2\rho^2 u = 0\,.
\end{aligned}
$$

Temos uma equação de Bessel modificada de ordem zero, com solução,

$$u(\rho) = c_1 I_0(\alpha\rho) + c_2 K_0(\alpha\rho)\,.$$

As condições de contorno nos dão,

$$
\begin{aligned}
c_1 I_0(\alpha a) + c_2 K_0(\alpha a) = u_a\,, \\
c_1 I_0(\alpha b) + c_2 K_0(\alpha b) = u_b\,,
\end{aligned}
$$

logo,

$$
\begin{aligned}
c_1 &= \frac{u_a K_0(\alpha b) - u_b K_0(\alpha a)}{I_0(\alpha a)K_0(\alpha b) - K_0(\alpha a)I_0(\alpha b)}\,, \\
c_2 &= \frac{u_b I_0(\alpha a) - u_a I_0(\alpha b)}{I_0(\alpha a)K_0(\alpha b) - K_0(\alpha a)I_0(\alpha b)}\,.
\end{aligned}
$$

A solução é portanto,

$$
\begin{aligned}
u(\rho) &= c_1 I_0(\alpha\rho) + c_2 K_0(\alpha\rho)\,, \\
&= u_a\frac{K_0(\alpha b)I_0(\alpha\rho) - I_0(\alpha b)K_0(\alpha\rho)}{I_0(\alpha a)K_0(\alpha b) - K_0(\alpha a)I_0(\alpha b)} \\
&\quad + u_b\frac{I_0(\alpha a)K_0(\alpha\rho) - K_0(\alpha a)I_0(\alpha\rho)}{I_0(\alpha a)K_0(\alpha b) - K_0(\alpha a)I_0(\alpha b)}\,.
\end{aligned}
$$

Usando,

$$V_n(\lambda_j\rho) \equiv I_n(\lambda_j\rho)K_n(\lambda_j b) - I_n(\lambda_j b)K_n(\lambda_j\rho)\,,$$

$$W_n(\lambda_j\rho) \equiv I_n(\lambda_j\rho)K_n(\lambda_j a) - I_n(\lambda_j a)K_n(\lambda_j\rho)\,,$$

temos,

$$u_p(\rho) = u_a \frac{V_0(\alpha\rho)}{V_0(\alpha a)} + u_b \frac{W_0(\alpha\rho)}{W_0(\alpha b)} \,.$$

(e) Agora temos a equação de Bessel de ordem zero,

$$\rho^2 \frac{d^2u}{d\rho^2} + \rho \frac{du}{d\rho} + \alpha^2 \rho^2 u = 0 \,,$$

com solução,

$$u(\rho) = c_1 J_0(\alpha\rho) + c_2 Y_0(\alpha\rho) \,.$$

As condições de contorno nos dão,

$$\begin{aligned} c_1 J_0(\alpha a) + c_2 Y_0(\alpha a) &= u_a \,, \\ c_1 J_0(\alpha b) + c_2 Y_0(\alpha b) &= u_b \,, \end{aligned}$$

logo,

$$\begin{aligned} c_1 &= \frac{u_a Y_0(\alpha b) - u_b Y_0(\alpha a)}{J_0(\alpha a) Y_0(\alpha b) - Y_0(\alpha a) J_0(\alpha b)} \,, \\ c_2 &= \frac{u_b J_0(\alpha a) - u_a J_0(\alpha b)}{J_0(\alpha a) Y_0(\alpha b) - Y_0(\alpha a) J_0(\alpha b)} \,. \end{aligned}$$

A solução é assim,

$$\begin{aligned} u(\rho) &= c_1 J_0(\alpha\rho) + c_2 Y_0(\alpha\rho) \,, \\ &= u_a \frac{Y_0(\alpha b) J_0(\alpha\rho) - J_0(\alpha b) Y_0(\alpha\rho)}{J_0(\alpha a) Y_0(\alpha b) - Y_0(\alpha a) J_0(\alpha b)} \\ &\quad + u_b \frac{J_0(\alpha a) Y_0(\alpha\rho) - Y_0(\alpha a) J_0(\alpha\rho)}{J_0(\alpha a) Y_0(\alpha b) - Y_0(\alpha a) J_0(\alpha b)} \,. \end{aligned}$$

Usando,

$$U_n(\lambda_j \rho) \equiv J_n(\lambda_j \rho) Y_n(\lambda_j a) - J_n(\lambda_j a) Y_n(\lambda_j \rho) \,,$$

$$Z_n(\lambda_j \rho) \equiv J_n(\lambda_j \rho) Y_n(\lambda_j b) - J_n(\lambda_j b) Y_n(\lambda_j \rho) \,,$$

temos,

$$u(\rho) = u_a \frac{Z_0(\alpha\rho)}{Z_0(\alpha a)} + u_b \frac{U_0(\alpha\rho)}{U_0(\alpha b)} \,.$$

3 Considerando $u = u(\rho, z)$

Substituindo $u(\rho, z) = R(\rho)Z(z)$ em (1), vem,

$$\frac{1}{\rho R}(\rho R')' + \frac{Z''}{Z} = 0\,,$$

em que $R' = dR/d\rho$, $Z' = dZ/dz$, etc. Da equação acima temos,

$$\frac{1}{\rho R}(\rho R')' = -\frac{Z''}{Z} \equiv -\lambda^2\,.$$

Portanto, obtemos as equações ordinárias,

$$\rho^2 R'' + \rho R' + (\lambda\rho)^2 R = 0\,, \tag{4}$$

$$Z'' - \lambda^2 Z = 0\,. \tag{5}$$

A solução da equação (5) é,

$$Z(z) = a_1 \text{cosh}\lambda z + a_2 \text{senh}\lambda z\,. \tag{6}$$

A equação (4) é a equação diferencial de Bessel de ordem zero, com solução,

$$R(\rho) = b_1 J_0(\lambda\rho) + b_2 Y_0(\lambda\rho)\,. \tag{7}$$

Notemos que as soluções acima são para $\lambda \neq 0$. Se $\lambda = 0$ as funções R e Z são constantes. Se $\lambda = 0$, temos de (4) e (5),

$$(\rho R')' = 0\,,$$
$$Z'' = 0\,.$$

As soluções das equações acima são,

$$R(\rho) = a_0 + b_0 \ln\rho\,, \tag{8}$$

$$Z(z) = c_0 z + d_0\,. \tag{9}$$

Pelo princípio da superposição, a solução geral é uma soma das possíveis soluções individuais,

$$\begin{aligned} u(\rho, z) &= (c_0 z + d_0)(a_0 + b_0 \ln\rho) \\ &\quad + \sum_{\lambda>0} [b_{1\lambda} J_0(\lambda\rho) + b_{2\lambda} Y_0(\lambda\rho)][a_{1\lambda} \text{cosh}\lambda z + a_{2\lambda} \text{senh}\lambda z]\,. \end{aligned} \tag{10}$$

A determinação das constantes e dos possíveis valores de λ_n depende das condições de contorno.

Consideremos agora a outra possibilidade para a constante de separação λ,

$$\frac{1}{\rho R}(\rho R')' = -\frac{Z''}{Z} \equiv +\lambda^2 \,.$$

Obtemos agora,

$$\rho^2 R'' + \rho R' - (\lambda\rho)^2 R = 0 \,, \quad (11)$$

$$Z'' + \lambda^2 Z = 0 \,. \quad (12)$$

A solução da equação para Z é,

$$Z(z) = a_1 \cos\lambda z + a_2 \text{sen}\,\lambda z \,. \quad (13)$$

A equação para R é a equação diferencial modificada de Bessel, com solução,

$$R(\rho) = b_1 I_0(\lambda\rho) + b_2 K_0(\lambda\rho) \,. \quad (14)$$

A solução geral agora é,

$$\begin{aligned} u(\rho, z) &= (c_0 z + d_0)(a_0 + b_0 \ln\rho) \\ &\quad + \sum_{\lambda>0} [b_{1\lambda} I_0(\lambda\rho) + b_{2\lambda} K_0(\lambda\rho)][a_{1\lambda}\cos\lambda z + a_{2\lambda}\text{sen}\,\lambda z] \,. \quad (15) \end{aligned}$$

4 Problemas

1. Considerando o intervalo $0 \leq \rho \leq a$, $0 \leq z \leq L$, encontre a solução $u(\rho, z)$ com as condições de contorno,

$$u(a, z) = f(z) \,,$$
$$u(\rho, 0) = g(\rho) \,,$$
$$u(\rho, L) = h(\rho) \,.$$

Escrevemos a solução como uma soma de três funções, cada uma satisfazendo uma das condições de contorno e se anulando nas outras,

$$u(\rho, z) = u_1(\rho, z) + u_2(\rho, z) + u_3(\rho, z) \,,$$

com,

$$
\begin{aligned}
&u_1(a,z) = f(z)\,, \quad u_1(\rho,0) = 0\,, \quad u_1(\rho,L) = 0\,,\\
&u_2(a,z) = 0\,, \quad u_2(\rho,0) = g(\rho)\,, \quad u_2(\rho,L) = 0\,,\\
&u_3(a,z) = 0\,, \quad u_3(\rho,0) = 0\,, \quad u_3(\rho,L) = h(\rho)\,.
\end{aligned}
$$

(a) Cálculo de u_1.
A solução geral finita em $\rho = 0$ é,

$$
\begin{aligned}
u_1(\rho,z) &= c_0 z + d_0 \\
&\quad + \sum_{\lambda>0} I_0(\lambda\rho)[a_{1\lambda}\cos\lambda z + a_{2\lambda} sen\,\lambda z]\,.
\end{aligned}
$$

Escrevendo as condições de contorno para u_1 temos,

$$
\begin{aligned}
&u_1(a,z) = f(z) = c_0 z + d_0 + \sum_{\lambda>0} I_0(\lambda a)[a_{1\lambda}\cos\lambda z + a_{2\lambda} sen\,\lambda z]\,,\\
&u_1(\rho,0) = 0 = d_0 + \sum_{\lambda>0} I_0(\lambda\rho) a_{1\lambda}\,,\\
&u_1(\rho,L) = 0 = c_0 L + d_0 + \sum_{\lambda>0} I_0(\lambda\rho)[a_{1\lambda}\cos\lambda L + a_{2\lambda} sen\,\lambda L]\,.
\end{aligned}
$$

Podemos satisfazer as duas últimas condições escolhendo,

$$
\begin{aligned}
&c_0 = d_0 = a_{1\lambda} = 0\,,\\
&\lambda_j L = \pi j\,, \quad j = 1, 2, \ldots
\end{aligned}
$$

A condição para $f(z)$ fica então,

$$
u_1(a,z) = f(z) = \sum_{j=1} I_0(\lambda_j a) a_{2j} sen\,(j\pi z/L)\,,
$$

que é a expansão de $f(z)$ em série de Fourier de senos, logo,

$$
I_0(\lambda_j a) a_{2j} = \frac{2}{L}\int_0^L f(x) sen\,(j\pi x/L)\,dx\,.
$$

A expansão em série de $f(z)$ é então, explicitamente,

$$f(z) = \frac{2}{L} \sum_{j=1} sen\,(j\pi z/L) \int_0^L f(x) sen\,(j\pi x/L)\,dx\,. \tag{16}$$

Incidentalmente, obtivemos uma representação para a função delta de Dirac no intervalo $(0, L)$. *Da definição*

$$f(z) = \int_0^L \delta(z-x) f(x) dx\,, \tag{17}$$

temos,

$$\delta(z-x) = \frac{2}{L} \sum_{j=1} sen\,(j\pi z/L) sen\,(j\pi x/L)\,. \tag{18}$$

Notemos que se $f(z) = f_0$ *constante, a série para* f *fica,*

$$\begin{aligned} f(z) &= \sum_{j=1} sen\,(j\pi z/L) \frac{2}{L} \int_0^L f(x) sen\,(j\pi x/L)\,dx\,, \\ f_0 &= f_0 \sum_{j=1} sen\,(j\pi z/L) \frac{2}{L} \int_0^L sen\,(j\pi x/L)\,dx\,, \\ f_0 &= f_0 \sum_{j=1} sen\,[(2j-1)\pi z/L] \frac{2}{L} \frac{2L}{(2j-1)\pi}\,, \\ f_0 &= \frac{4 f_0}{\pi} \sum_{j=1} \frac{sen\,[(2j-1)\pi z/L]}{2j-1}\,, \end{aligned}$$

em que usamos,

$$\begin{aligned} \int_0^L sen\,(j\pi x/L)\,dx &= -\frac{L}{j\pi} \cos(j\pi x/L)\Big|_0^L = -\frac{L}{j\pi} \cos j\pi + \frac{L}{j\pi}\,, \\ &= \frac{L}{j\pi}[1 - \cos j\pi] = \frac{L}{j\pi}[1 - (-1)^j]\,, \\ &= \begin{cases} 0\,, & j\ par\,, \\ \dfrac{2L}{j\pi}\,, & j\ \acute{i}mpar. \end{cases} \\ &= \frac{2L}{(2j-1)\pi}\,, j = 1, 2, 3, \ldots \end{aligned} \tag{19}$$

Vemos que a expansão de 1 no intervalo $0 \leq z \leq L$ *é,*

$$1 = \frac{4}{\pi}\sum_{j=1}\frac{sen\,[(2j-1)\pi z/L]}{2j-1}\,. \tag{20}$$

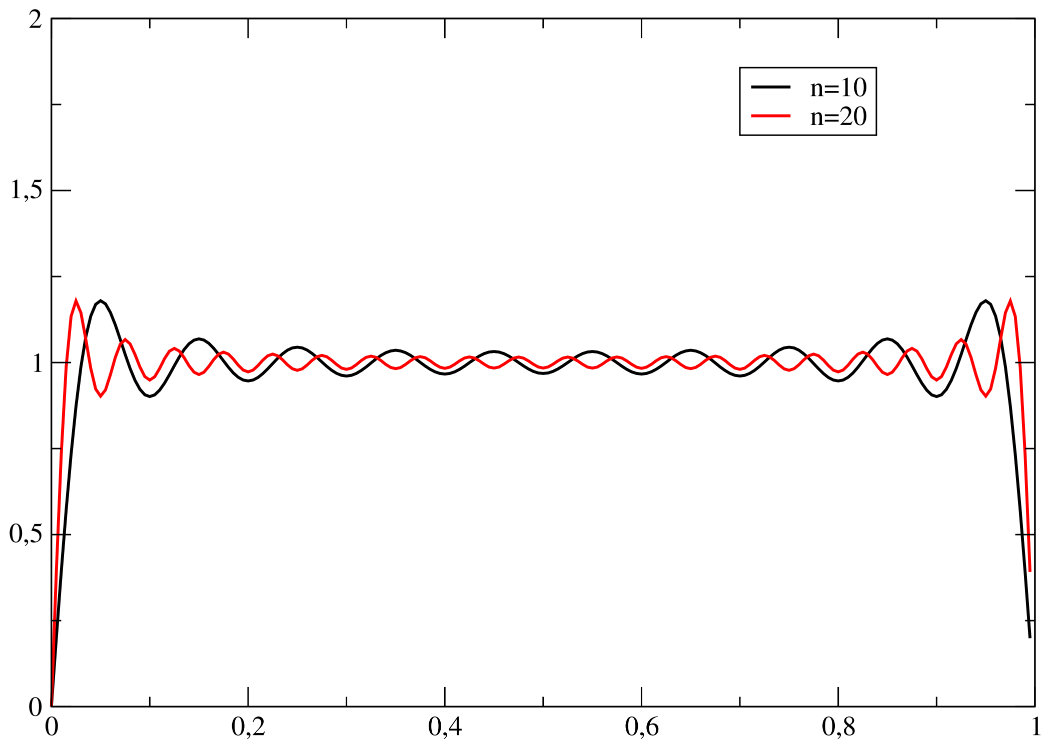

Fig. 1. Expansão de 1 em $0 \leq z \leq L$ para $L = 1$. O número de termos na soma é indicado.

A solução u_1 é então, para $f(z)$ qualquer,

$$u_1(\rho, z) = \sum_{j=1}\frac{I_0(\lambda_j\rho)}{I_0(\lambda_j a)}sen\,\lambda_j z\frac{2}{L}\int_0^L f(x)sen\,(j\pi x/L)\,dx\,.$$

Podemos escrever a solução acima como,

$$u_1 = \int_0^L f(x)G(x,\rho,z)dx\,,$$

com,

$$G(x,\rho,z) = \frac{2}{L}\sum_{j=1}\frac{I_0(\lambda_j\rho)}{I_0(\lambda_j a)}sen\,(\lambda_j z)sen\,(\lambda_j x)\,,\;\; \lambda_j = j\pi/L\,.$$

Se $f = f_0$ constante ([7], probl. 6.101),

$$
\begin{aligned}
u_1(\rho,z) &= \sum_{j=1} \frac{I_0(\lambda_j\rho)}{I_0(\lambda_j a)} sen\,\lambda_j z \frac{2}{L}\int_0^L f(x) sen\,(j\pi x/L)\,dx\,,\\
&= \sum_{j=1} \frac{I_0(\lambda_j\rho)}{I_0(\lambda_j a)} sen\,\lambda_j z \frac{2}{L} f_0 \int_0^L sen\,(j\pi x/L)\,dx\,,\\
&= \sum_{j=1} \frac{I_0(\lambda_j\rho)}{I_0(\lambda_j a)} sen\,\lambda_j z \frac{2f_0}{j\pi}(1-\cos j\pi)\,,\\
&= \frac{4f_0}{\pi}\sum_{j=1} \frac{I_0((2j-1)\pi\rho/L)}{I_0((2j-1)\pi a/L)} \frac{sen\,[(2j-1)\pi z/L]}{2j-1}\,.
\end{aligned}
$$

Se $f_0 = 0$ temos $u_1 = 0$.

(b) Cálculo de u_2.

Escrevemos a solução geral finita em $\rho = 0$ de forma conveniente como,

$$
u_2(\rho,z) = c_0 z + d_0 + \sum_{\lambda>0} J_0(\lambda\rho) a_{2\lambda} senh\,\lambda(z-L)\,.
$$

Escrevendo as condições de contorno para u_2 temos,

$$
\begin{aligned}
&u_2(a,z) = 0 = c_0 z + d_0 + \sum_{\lambda>0} J_0(\lambda a) a_{2\lambda} senh\,\lambda(z-L)\,,\\
&u_2(\rho,0) = g(\rho) = d_0 - \sum_{\lambda>0} J_0(\lambda\rho) a_{2\lambda} senh\,\lambda L\,,\\
&u_2(\rho,L) = 0 = c_0 L + d_0\,.
\end{aligned}
$$

Podemos satisfazer a primeira e a última condições escolhendo,

$$
\begin{aligned}
&c_0 = d_0 = 0\,,\\
&J_0(\lambda_j a) = 0\,,\quad j = 1,2,\ldots
\end{aligned}
$$

A segunda condição é então a expansão de $g(\rho)$ em funções de Bessel, logo,

$$
a_{2j} = -\frac{2}{a^2 J_1^2(\lambda_j a) senh\,\lambda_j L}\int_0^a x J_0(\lambda_j x) g(x) dx\,.
$$

Explicitamentem a expansão de $g(\rho)$ é então,

$$g(\rho) = \frac{2}{a^2} \sum_j \frac{J_0(\lambda_j \rho)}{J_1^2(\lambda_j a)} \int_0^a x J_0(\lambda_j x) g(x) dx\,, \quad J_0(\lambda_j a) = 0\,. \tag{21}$$

Se $g(\rho) = g_0$ *constante temos,*

$$\begin{aligned}
g(\rho) &= \frac{2}{a^2} \sum_j \frac{J_0(\lambda_j \rho)}{J_1^2(\lambda_j a)} \int_0^a x J_0(\lambda_j x) g(x) dx\,, \\
g_0 &= g_0 \frac{2}{a^2} \sum_j \frac{J_0(\lambda_j \rho)}{J_1^2(\lambda_j a)} \int_0^a x J_0(\lambda_j x) dx\,, \\
g_0 &= g_0 \frac{2}{a^2} \sum_j \frac{J_0(\lambda_j \rho)}{J_1^2(\lambda_j a)} \frac{a}{\lambda_j} J_1(\lambda_j a)\,, \\
g_0 &= g_0 \frac{2}{a} \sum_j \frac{J_0(\lambda_j \rho)}{\lambda_j J_1(\lambda_j a)}\,,
\end{aligned}$$

em que usamos

$$\int_0^a x J_0(\lambda x) dx = \frac{a}{\lambda} J_1(\lambda a)\,. \tag{22}$$

Da expressão acima temos uma expansão para 1 [7],

$$1 = \frac{2}{a} \sum_{j=1} \frac{J_0(\lambda_j \rho)}{\lambda_j J_1(\lambda_j a)}\,, \quad J_0(\lambda_j a) = 0\,. \tag{23}$$

É interessante fazer o gráfico da expressão acima para alguns termos da série. As primeiras raízes de $J_0(\lambda_j a) = 0$ *são,*

$$\lambda_j a = 2,4048,\ 5,5201,\ 8,6537,\ 11,7915,\ 14,9309,\ 18,0711, \ldots$$

A figura 2 mostra a série em (23) para alguns termos na soma.

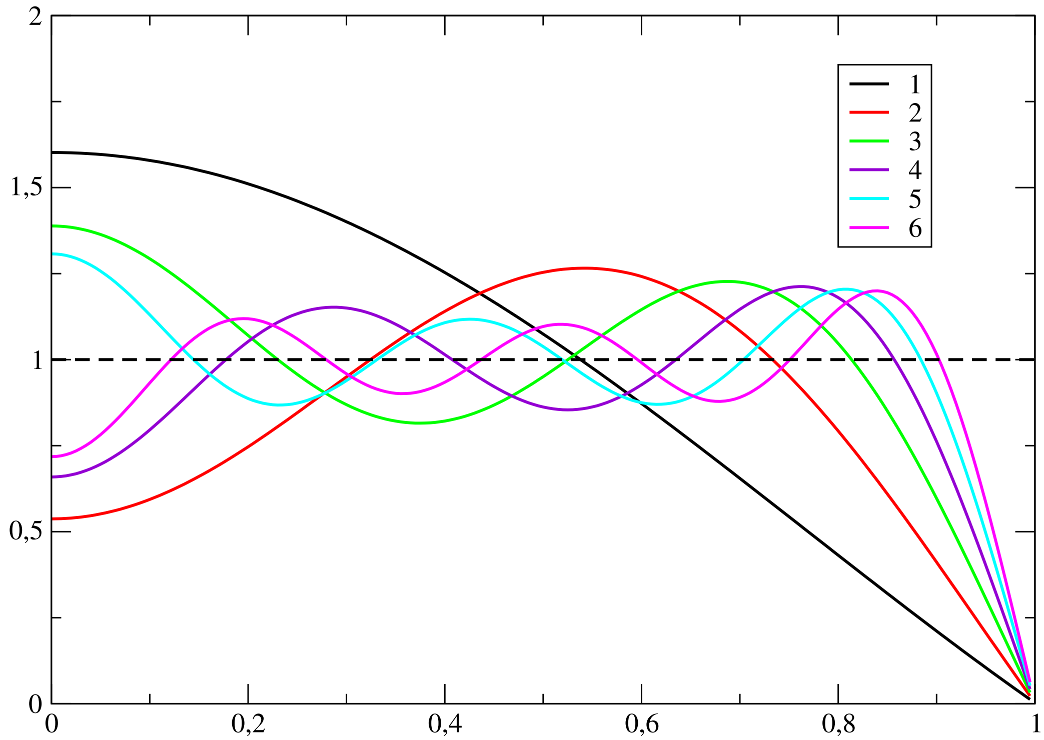

Fig. 2. Série para a função constante 1 em $0 \leq \rho \leq a$, na equação (23), para alguns termos na série. Usamos $a = 1$.

A expressão (21) para a expansão de $g(\rho)$ nos dá uma representação para a função delta de Dirac no intervalo $0 \leq \rho \leq a$. Da equação,

$$g(\rho) = \int_0^a \delta(\rho - x) g(x) dx \,,$$

temos,

$$\delta(\rho - x) = \frac{2}{a^2} \sum_j \frac{J_0(\lambda_j \rho)}{J_1^2(\lambda_j a)} x J_0(\lambda_j x) \,, \quad J_0(\lambda_j a) = 0 \,. \tag{24}$$

A solução u_2 é então ([7], prob. 6.93),

$$u_2(\rho, z) = -\frac{2}{a^2} \sum_{j=1} \frac{senh\, \lambda_j (z - L)}{senh\, \lambda_j L} \frac{J_0(\lambda_j \rho)}{J_1^2(\lambda_j a)} \int_0^a x J_0(\lambda_j x) g(x) dx \,.$$

Podemos escrever a solução acima como,

$$u_2(\rho, z) = \int_0^a G(x, \rho, z) g(x) dx \,,$$

com,

$$G(x,\rho,z) = -\frac{2}{a^2}\sum_{j=1}\frac{senh\,\lambda_j(z-L)}{senh\,\lambda_j L}\frac{J_0(\lambda_j\rho)}{J_1^2(\lambda_j a)}xJ_0(\lambda_j x)\,, \quad J_0(\lambda_j a)=0\,.$$

Em $z=0$,

$$u_2(\rho,0) = \sum_{j=1}\frac{2J_0(\lambda_j\rho)}{a^2J_1^2(\lambda_j a)}\int_0^a xJ_0(\lambda_j x)g(x)dx = g(\rho)\,,$$

como esperado. Se $g(\rho)=g_0$ *constante temos,*

$$\begin{aligned} u_2(\rho,z) &= -\sum_{j=1}\frac{senh\,\lambda_j(z-L)}{senh\,\lambda_j L}\frac{2J_0(\lambda_j\rho)}{a^2J_1^2(\lambda_j a)}g_0\int_0^a xJ_0(\lambda_j x)dx\,, \\ &= -\frac{2g_0}{a}\sum_{j=1}\frac{senh\,\lambda_j(z-L)}{\lambda_j senh\,\lambda_j L}\frac{J_0(\lambda_j\rho)}{J_1(\lambda_j a)}\,. \end{aligned}$$

Em $z=0$,

$$u_2(\rho,0) = \frac{2g_0}{a}\sum_{j=1}\frac{J_0(\lambda_j\rho)}{\lambda_j J_1(\lambda_j a)} = g_0\,,$$

como esperado. Se $g_0=0$ *temos* $u_2=0$.

(c) Cálculo de u_3.

Escrevemos a solução geral finita em $\rho=0$ *como,*

$$u_3(\rho,z) = c_0z + d_0 + \sum_{\lambda>0} J_0(\lambda\rho)a_{2\lambda}senh\,\lambda z\,.$$

Escrevendo as condições de contorno para u_3 *temos,*

$$\begin{aligned} &u_3(a,z) = 0 = c_0z + d_0 + \sum_{\lambda>0}J_0(\lambda a)a_{2\lambda}senh\,\lambda z\,, \\ &u_3(\rho,0) = 0 = d_0\,, \\ &u_3(\rho,L) = h(\rho) = c_0L + d_0 + \sum_{\lambda>0}J_0(\lambda\rho)a_{2\lambda}senh\,\lambda L\,. \end{aligned}$$

Podemos satisfazer as duas primeiras condições escolhendo,

$$c_0 = d_0 = 0\,,$$
$$J_0(\lambda_j a) = 0\,, \quad j = 1, 2, \ldots$$

A terceira condição é então a expansão de $h(\rho)$ em funções de Bessel, logo,

$$a_{2j} = \frac{2}{a^2 J_1^2(\lambda_j a) senh\, \lambda_j L} \int_0^a x J_0(\lambda_j x) h(x) dx\,, \quad J_0(\lambda_j a) = 0\,.$$

A expansão de $h(\rho)$ é assim,

$$h(\rho) = \frac{2}{a^2} \sum_{j=1} \frac{J_0(\lambda_j \rho)}{J_1^2(\lambda_j a)} \int_0^a x J_0(\lambda_j x) h(x) dx\,.$$

A solução u_3 é então,

$$u_3(\rho, z) = \frac{2}{a^2} \sum_{j=1} \frac{senh\, \lambda_j z}{senh\, \lambda_j L} \frac{J_0(\lambda_j \rho)}{J_1^2(\lambda_j a)} \int_0^a x J_0(\lambda_j x) h(x) dx\,.$$

Podemos escrever a solução acima como,

$$u_3(\rho, z) = \int_0^a G(x, \rho, z) h(x) dx\,,$$

com,

$$G(x, \rho, z) = \frac{2}{a^2} \sum_{j=1} \frac{senh\, \lambda_j z}{senh\, \lambda_j L} \frac{J_0(\lambda_j \rho)}{J_1^2(\lambda_j a)} x J_0(\lambda_j x)\,.$$

Se $h = h_0$ constante ([7], probl. 6.111),

$$\begin{aligned} u_3(\rho, z) &= \sum_{j=1} \frac{senh\, \lambda_j z}{senh\, \lambda_j L} \frac{2 J_0(\lambda_j \rho)}{a^2 J_1^2(\lambda_j a)} h_0 \int_0^a x J_0(\lambda_j x) dx\,, \\ &= \frac{2h_0}{a} \sum_{j=1} \frac{senh\, \lambda_j z}{\lambda_j senh\, \lambda_j L} \frac{J_0(\lambda_j \rho)}{J_1(\lambda_j a)}\,, \end{aligned}$$

em que usamos $xJ_0(x) = (xJ_1(x))'$. Se $h_0 = 0$ temos $u_3 = 0$.

(d) Se as funções nas condições de contorno são constantes, isto é,

$$u(a, z) = f_0\,,$$
$$u(\rho, 0) = g_0\,,$$
$$u(\rho, L) = h_0\,,$$

a solução é,

$$
\begin{aligned}
u(\rho,z) \;=\; & \frac{4f_0}{\pi}\sum_{j=1}\frac{I_0((2j-1)\pi\rho/L)}{I_0((2j-1)\pi a/L)}\frac{sen\,[(2j-1)\pi z/L]}{2j-1} \\
& -\frac{2g_0}{a}\sum_{j=1}\frac{senh\,\lambda_j(z-L)}{\lambda_j senh\,\lambda_j L}\frac{J_0(\lambda_j\rho)}{J_1(\lambda_j a)} \\
& +\frac{2h_0}{a}\sum_{j=1}\frac{senh\,\lambda_j z}{\lambda_j senh\,\lambda_j L}\frac{J_0(\lambda_j\rho)}{J_1(\lambda_j a)}\,, \quad J_0(\lambda_j a)=0\,.
\end{aligned}
$$

2. Considerando o intervalo $0 \le \rho \le a$, $0 \le z \le L$, encontre a solução $u(\rho, z)$ com as condições de contorno,

$$
\begin{aligned}
\frac{\partial u(a,z)}{\partial\rho} &= f(z)\,, \\
u(\rho,0) &= g(\rho)\,, \\
u(\rho,L) &= h(\rho)\,.
\end{aligned}
$$

Escrevemos a solução como uma soma de três funções, cada uma satisfazendo uma das condições de contorno e se anulando nas outras,

$$
u(\rho,z) = u_1(\rho,z) + u_2(\rho,z) + u_3(\rho,z)\,,
$$

com,

$$
\begin{aligned}
\frac{\partial u_1(a,z)}{\partial\rho} &= f(z)\,, \;\; u_1(\rho,0)=0\,, \;\; u_1(\rho,L)=0\,, \\
\frac{\partial u_2(a,z)}{\partial\rho} &= 0\,, \;\; u_2(\rho,0)=g(\rho)\,, \;\; u_2(\rho,L)=0\,, \\
\frac{\partial u_3(a,z)}{\partial\rho} &= 0\,, \;\; u_3(\rho,0)=0\,, \;\; u_3(\rho,L)=h(\rho)\,.
\end{aligned}
$$

(a) Cálculo de u_1. A solução geral finita em $\rho = 0$ é,

$$
\begin{aligned}
u_1(\rho,z) \;=\; & c_0 z + d_0 \\
& +\sum_{\lambda>0} I_0(\lambda\rho)[a_{1\lambda}\cos\lambda z + a_{2\lambda} sen\,\lambda z]\,.
\end{aligned}
$$

Escrevendo as condições de contorno para u_1 temos,

$$\begin{aligned}
\frac{\partial u_1(a,z)}{\partial \rho} &= f(z) = \sum_{\lambda>0} \lambda I_1(\lambda a)[a_{1\lambda}\cos\lambda z + a_{2\lambda} sen\,\lambda z]\,,\\
u_1(\rho,0) &= 0 = d_0 + \sum_{\lambda>0} I_0(\lambda\rho)a_{1\lambda}\,,\\
u_1(\rho,L) &= 0 = c_0 L + d_0\\
&\quad + \sum_{\lambda>0} I_0(\lambda\rho)[a_{1\lambda}\cos\lambda L + a_{2\lambda} sen\,\lambda L]\,,
\end{aligned}$$

em que usamos $I_0'(x) = I_1(x)$. Podemos satisfazer as duas últimas condições escolhendo,

$$\begin{aligned}
&c_0 = d_0 = 0\,,\\
&a_{1\lambda} = 0\,,\\
&\lambda_j L = \pi j\,,\quad j = 1, 2, \ldots
\end{aligned}$$

A primeira condição fica então,

$$\frac{\partial u_1(a,z)}{\partial \rho} = f(z) = \sum_{j=1} \lambda_j I_1(\lambda_j a) a_{2j} sen\,(j\pi z/L)\,,$$

que é a expansão de $f(z)$ em série de Fourier de senos, logo,

$$\lambda_j I_1(\lambda_j a) a_{2j} = \frac{2}{L}\int_0^L f(x) sen\,(j\pi x/L)\,dx\,,$$

ou,

$$a_{2j} = \frac{1}{\lambda_j I_1(\lambda_j a)}\frac{2}{L}\int_0^L f(x) sen\,(j\pi x/L)\,dx\,.$$

A expansão de $f(z)$ fica,

$$f(z) = \sum_{j=1} sen\,(j\pi z/L)\frac{2}{L}\int_0^L f(x) sen\,(j\pi x/L)\,dx\,.$$

A solução u_1 é então,

$$u_1(\rho,z) = \frac{2}{L}\sum_{j=1}\frac{I_0(\lambda_j\rho)}{\lambda_j I_1(\lambda_j a)} sen\,(j\pi z/L)\int_0^L f(x) sen\,(j\pi x/L)\,dx\,.$$

Podemos escrever a solução acima como,

$$u_1(\rho, z) = \int_0^L G(x, \rho, z) f(x)\, dx\,,$$

com,

$$G(x, \rho, z) = \frac{2}{L} \sum_{j=1} \frac{I_0(\lambda_j \rho)}{\lambda_j I_1(\lambda_j a)} sen\,(j\pi z/L) sen\,(j\pi x/L)\,.$$

Se $f = f_0$ constante,

$$\begin{aligned} u_1(\rho, z) &= \frac{2}{L} \sum_{j=1} \frac{I_0(\lambda_j \rho)}{\lambda_j I_1(\lambda_j a)} sen\,(j\pi z/L) f_0 \int_0^L sen\,(j\pi x/L)\, dx\,, \\ &= \frac{4Lf_0}{\pi^2} \sum_{j=1} \frac{I_0((2j-1)\pi\rho/L)}{(2j-1) I_1((2j-1)\pi a/L)} \frac{sen\,[(2j-1)\pi z/L]}{2j-1}\,. \end{aligned}$$

Se $f_0 = 0$ temos $u_1 = 0$.

(b) Cálculo de u_2.

Escrevemos u_2 como,

$$u_2(\rho, z) = c_0 z + d_0 + \sum_{\lambda > 0} J_0(\lambda\rho) a_{2\lambda} senh\,\lambda(z - L)\,.$$

As condições de contorno ficam,

$$\begin{aligned} &\frac{\partial u_2(a, z)}{\partial \rho} = 0 = -\sum_{\lambda>0} \lambda J_1(\lambda a) a_{2\lambda} senh\,\lambda(z - L)\,, \\ &u_2(\rho, 0) = g(\rho) = d_0 - \sum_{\lambda>0} J_0(\lambda\rho) a_{2\lambda} senh\,\lambda L\,, \\ &u_2(\rho, L) = 0 = c_0 L + d_0\,, \end{aligned}$$

em que usamos $J_0'(x) = -J_1(x)$. Satisfazemos as condições acima escolhendo,

$$\begin{aligned} &d_0 = -c_0 L\,, \\ &J_1(\lambda_j a) = 0\,, \quad j = 1, 2, \ldots \end{aligned}$$

A condição para $g(\rho)$ é uma série em funções de Bessel,

$$g(\rho) = d_0 - \sum_{j=1} J_0(\lambda_j \rho) a_{2j} senh\, \lambda_j L\,, \quad J_1(\lambda_j a) = 0\,,$$

logo,

$$\begin{aligned} d_0 &= \frac{2}{a^2} \int_0^a x g(x) dx\,, \\ -a_{2j} senh\, \lambda_j L &= \frac{2}{a^2 J_0^2(\lambda_k a)} \int_0^a x J_0(\lambda_k x) g(x) dx\,, \end{aligned}$$

ou,

$$a_{2j} = -\frac{2}{a^2 J_0^2(\lambda_j a) senh\, \lambda_j L} \int_0^a x J_0(\lambda_j x) g(x) dx\,.$$

Explicitamente,

$$g(\rho) = \frac{2}{a^2} \int_0^a x g(x) dx + \frac{2}{a^2} \sum_{j=1} \frac{J_0(\lambda_j \rho)}{J_0^2(\lambda_j a)} \int_0^a x J_0(\lambda_j x) g(x) dx\,, \quad J_1(\lambda_j a) = 0\,.$$

Se $g(\rho) = g_0$ *constante,*

$$\begin{aligned} g(\rho) &= \frac{2}{a^2} \int_0^a x g(x) dx + \frac{2}{a^2} \sum_{j=1} \frac{J_0(\lambda_j \rho)}{J_0^2(\lambda_j a)} \int_0^a x J_0(\lambda_j x) g(x) dx\,, \\ g_0 &= \frac{2}{a^2} g_0 \int_0^a x dx + \frac{2}{a^2} \sum_{j=1} \frac{J_0(\lambda_j \rho)}{J_0^2(\lambda_j a)} g_0 \int_0^a x J_0(\lambda_j x) dx\,, \\ g_0 &= \frac{2}{a^2} g_0 \frac{a^2}{2} + \frac{2}{a^2} \sum_{j=1} \frac{J_0(\lambda_j \rho)}{J_0^2(\lambda_j a)} g_0 \frac{a}{\lambda_j} J_1(\lambda_j a)\,, \\ g_0 &= g_0\,, \end{aligned}$$

como esperado.

A solução u_2 *é então,*

$$\begin{aligned} u_2(\rho, z) &= c_0 z + d_0 + \sum_{j=1} J_0(\lambda_j \rho) a_{2j} senh\, \lambda_j (z - L)\,, \\ &= \left(1 - \frac{z}{L}\right) \frac{2}{a^2} \int_0^a x g(x) dx \\ &\quad - \frac{2}{a^2} \sum_{j=1} \frac{J_0(\lambda_j \rho) senh\, \lambda_j (z - L)}{J_0^2(\lambda_j a) senh\, \lambda_j L} \int_0^a x J_0(\lambda_j x) g(x) dx\,. \end{aligned}$$

Se $g(\rho) = g_0$ *constante,*

$$\begin{aligned}
u_2(\rho, z) &= \left(1 - \frac{z}{L}\right) \frac{2}{a^2} \int_0^a x g(x) dx \\
&\quad - \frac{2}{a^2} \sum_{j=1} \frac{J_0(\lambda_j \rho) senh\, \lambda_j (z - L)}{J_0^2(\lambda_j a) senh\, \lambda_j L} \int_0^a x J_0(\lambda_j x) g(x) dx \,, \\
&= \left(1 - \frac{z}{L}\right) \frac{2}{a^2} g_0 \int_0^a x dx \\
&\quad - \frac{2}{a^2} \sum_{j=1} \frac{J_0(\lambda_j \rho) senh\, \lambda_j (z - L)}{J_0^2(\lambda_j a) senh\, \lambda_j L} g_0 \int_0^a x J_0(\lambda_j x) dx \,, \\
&= \left(1 - \frac{z}{L}\right) g_0 \\
&\quad - \frac{2 g_0}{a} \sum_{j=1} \frac{J_0(\lambda_j \rho) senh\, \lambda_j (z - L)}{\lambda_j J_0^2(\lambda_j a) senh\, \lambda_j L} J_1(\lambda_j a) \,, \\
&= \left(1 - \frac{z}{L}\right) g_0 \,,
\end{aligned}$$

pois $J_1(\lambda_j a) = 0$. *Se* $g_0 = 0$ *temos* $u_2 = 0$.

(c) *Cálculo de* u_3.

Escrevemos u_3 *como,*

$$u_3(\rho, z) = c_0 z + d_0 + \sum_{\lambda > 0} J_0(\lambda \rho) a_{2\lambda} senh\, \lambda z \,.$$

As condições de contorno nos dão,

$$\begin{aligned}
&\frac{\partial u_3(a, z)}{\partial \rho} = 0 = - \sum_{\lambda > 0} \lambda J_1(\lambda a) a_{2\lambda} senh\, \lambda z \,, \\
&u_3(\rho, 0) = 0 = d_0 \,, \\
&u_3(\rho, L) = h(\rho) = c_0 L + d_0 + \sum_{\lambda > 0} J_0(\lambda \rho) a_{2\lambda} senh\, \lambda L \,.
\end{aligned}$$

Satisfazemos as condições acima escolhendo,

$$\begin{aligned}
&J_1(\lambda_j a) = 0 \,, \quad j = 1, 2, \ldots \\
&d_0 = 0 \,.
\end{aligned}$$

A condição para $h(\rho)$ *é assim,*

$$h(\rho) = c_0 L + \sum_{j=1} J_0(\lambda_j \rho) a_{2j} senh\, \lambda_j L\,,$$

logo,

$$c_0 L = \frac{2}{a^2} \int_0^a x h(x) dx\,,$$
$$a_{2k} senh\, \lambda_k L = \frac{2}{a^2 J_0^2(\lambda_k a)} \int_0^a x J_0(\lambda_k x) h(x) dx\,,$$

ou,

$$c_0 = \frac{2}{a^2 L} \int_0^a x h(x) dx\,,$$
$$a_{2k} = \frac{2}{a^2 J_0^2(\lambda_k a) senh\, \lambda_k L} \int_0^a x J_0(\lambda_k x) h(x) dx\,.$$

A expansão para $h(\rho)$ *é assim,*

$$\begin{aligned} h(\rho) &= \frac{2}{a^2} \int_0^a x h(x) dx \\ &+ \frac{2}{a^2} \sum_{j=1} \frac{J_0(\lambda_j \rho)}{J_0^2(\lambda_j a)} \int_0^a x J_0(\lambda_j x) h(x) dx\,, \quad J_1(\lambda_j a) = 0\,. \end{aligned}$$

Se $h(\rho) = h_0$*,*

$$\begin{aligned} h(\rho) &= \frac{2}{a^2} \int_0^a x h(x) dx \\ &+ \sum_{j=1} J_0(\lambda_j \rho) \frac{2}{a^2 J_0^2(\lambda_j a)} \int_0^a x J_0(\lambda_j x) h(x) dx\,, \\ h_0 &= \frac{2}{a^2} h_0 \int_0^a x dx \\ &+ \sum_{j=1} J_0(\lambda_j \rho) \frac{2}{a^2 J_0^2(\lambda_j a)} h_0 \int_0^a x J_0(\lambda_j x) dx\,, \\ h_0 &= h_0\,, \end{aligned}$$

como esperado.

A solução u_3 *é então,*

$$
\begin{aligned}
u_3(\rho, z) &= \frac{2z}{a^2 L}\int_0^a xh(x)dx \\
&+\frac{2}{a^2}\sum_{j=1}\frac{J_0(\lambda_j\rho) senh\,\lambda_j z}{J_0^2(\lambda_j a) senh\,\lambda_j L}\int_0^a xJ_0(\lambda_j x)h(x)dx\,.
\end{aligned}
$$

Se $h(\rho) = h_0$ constante,

$$
\begin{aligned}
u_3(\rho, z) &= \frac{2z}{a^2 L}\int_0^a xh(x)dx \\
&+\frac{2}{a^2}\sum_{j=1}\frac{J_0(\lambda_j\rho) senh\,\lambda_j z}{J_0^2(\lambda_j a) senh\,\lambda_j L}\int_0^a xJ_0(\lambda_j x)h(x)dx\,, \\
&= \frac{2z}{a^2 L}h_0\int_0^a xdx \\
&+\frac{2}{a^2}\sum_{j=1}\frac{J_0(\lambda_j\rho) senh\,\lambda_j z}{J_0^2(\lambda_j a) senh\,\lambda_j L}h_0\int_0^a xJ_0(\lambda_j x)dx\,, \\
&= \frac{z}{L}h_0 \\
&+\frac{2}{a^2}\sum_{j=1}\frac{J_0(\lambda_j\rho) senh\,\lambda_j z}{J_0^2(\lambda_j a) senh\,\lambda_j L}h_0\frac{a}{\lambda_j}J_1(\lambda_j a)\,, \\
&= \frac{z}{L}h_0\,.
\end{aligned}
$$

Se $h_0 = 0$ temos $u_3 = 0$.

(d) Se as condições de contorno são todas dadas por constantes temos,

$$
\begin{aligned}
u(\rho, z) &= \sum_{j=1}\frac{I_0(\lambda_j\rho)}{\lambda_j I_1(\lambda_j a)} sen\,(j\pi z/L)\frac{2}{L}f_0\int_0^L sen\,(j\pi x/L)\,dx\,, \\
&= \frac{4Lf_0}{\pi^2}\sum_{j=1}\frac{I_0((2j-1)\pi\rho/L)}{(2j-1)I_1((2j-1)\pi a/L)}\frac{sen\,[(2j-1)\pi z/L]}{2j-1} \\
&+\left(1-\frac{z}{L}\right)g_0 + \frac{z}{L}h_0\,.
\end{aligned}
$$

3. Considerando o intervalo $0 \leq \rho \leq a$, $0 \leq z \leq L$, encontre a solução $u(\rho, z)$ com as condições de contorno,

$$
\begin{aligned}
u(a,z) &= f(z)\,, \\
\frac{\partial u(\rho,0)}{\partial z} &= g(\rho)\,, \\
u(\rho,L) &= h(\rho)\,.
\end{aligned}
$$

Escrevemos a solução como uma soma de três funções, cada uma satisfazendo uma das condições de contorno e se anulando nas outras,

$$u(\rho,z) = u_1(\rho,z) + u_2(\rho,z) + u_3(\rho,z)\,,$$

com,

$$
\begin{aligned}
u_1(a,z) = f(z)\,, \quad \frac{\partial u_1(\rho,0)}{\partial z} = 0\,, \quad u_1(\rho,L) = 0\,, \\
u_2(a,z) = 0\,, \quad \frac{\partial u_2(\rho,0)}{\partial z} = g(\rho)\,, \quad u_2(\rho,L) = 0\,, \\
u_3(a,z) = 0\,, \quad \frac{\partial u_3(\rho,0)}{\partial z} = 0\,, \quad u_3(\rho,L) = h(\rho)\,.
\end{aligned}
$$

(a) Cálculo de u_1.
Escrevemos u_1 na forma,

$$
\begin{aligned}
u_1(\rho,z) &= c_0 z + d_0 \\
&\quad + \sum_{\lambda>0} I_0(\lambda\rho)[a_{1\lambda}\cos\lambda z + a_{2\lambda} sen\,\lambda z]\,.
\end{aligned}
$$

As condições de contorno nos dão,

$$
\begin{aligned}
u_1(a,z) &= f(z) = c_0 z + d_0 \\
&\quad + \sum_{\lambda>0} I_0(\lambda a)[a_{1\lambda}\cos\lambda z + a_{2\lambda} sen\,\lambda z]\,, \\
\frac{\partial u_1(\rho,0)}{\partial z} &= 0 = c_0 + \sum_{\lambda>0} I_0(\lambda\rho)\lambda a_{2\lambda}\,, \\
u_1(\rho,L) &= 0 = c_0 L + d_0 \\
&\quad + \sum_{\lambda>0} I_0(\lambda\rho)[a_{1\lambda}\cos\lambda L + a_{2\lambda} sen\,\lambda L]\,.
\end{aligned}
$$

Satisfazemos as condições acima escolhendo,

$$c_0 = d_0 = 0\,,$$
$$a_{2\lambda} = 0\,,$$
$$\lambda_j L = \frac{\pi}{2}(2j-1)\,,\quad j = 1, 2, \ldots$$

A condição para $f(z)$ é assim,

$$f(z) \;=\; \sum_{j=1} I_0((2j-1)\pi a/2L)a_{1j}\cos[(2j-1)\pi z/2L]\,.$$

Multiplicando a expressão acima por $\cos[(2k-1)\pi z/2L]$ *e integrando de* 0 *a* $+L$*,*

$$\int_0^{+L} f(z)\cos[(2k-1)\pi z/2L]dz = \sum_{j=1} I_0((2j-1)\pi a/2L)a_{1j}\,\times$$
$$\times\int_0^{+L}\cos[(2j-1)\pi z/2L]\cos[(2k-1)\pi z/2L]dz\,,$$
$$= I_0((2k-1)\pi a/2L)a_{1k}\frac{L}{2}\,.$$

Usamos a identidade,

$$\int_0^{+L}\cos[(2j-1)\pi z/2L]\cos[(2k-1)\pi z/2L]dz = \begin{cases} 0\,, j \neq k\,, \\ \dfrac{L}{2}\,, j = k\,. \end{cases}$$

Portanto,

$$a_{1k} = \frac{2}{LI_0((2k-1)\pi a/2L)}\int_0^{+L} f(z)\cos[(2k-1)\pi z/2L]dz\,.$$

A expansão para $f(z)$ é então,

$$f(z) = \frac{2}{L}\sum_{j=1}\cos[(2j-1)\pi z/2L]\int_0^{+L} f(x)\cos[(2j-1)\pi x/2L]dx\,. \qquad (25)$$

Obtemos assim outra representação para a função delta de Dirac no intervalo $0 \leq z \leq +L$*,*

$$\delta(z-x) = \frac{2}{L}\sum_{j=1} \cos[(2j-1)\pi z/2L]\cos[(2j-1)\pi x/2L]\,, \tag{26}$$

Se $f(z) = f_0$ constante,

$$\begin{aligned}
f(z) &= \frac{2}{L}\sum_{j=1}\cos[(2j-1)\pi z/2L]\int_0^{+L} f(x)\cos[(2j-1)\pi x/2L]dx\,, \\
f_0 &= f_0\frac{2}{L}\sum_{j=1}\cos[(2j-1)\pi z/2L]\int_0^{+L}\cos[(2j-1)\pi x/2L]dx\,, \\
f_0 &= f_0\frac{2}{L}\sum_{j=1}\cos[(2j-1)\pi z/2L]\frac{2L}{(2j-1)\pi}sen\,[(2j-1)\pi/2]\,, \\
f_0 &= f_0\frac{4}{\pi}\sum_{j=1}\frac{1}{2j-1}\cos[(2j-1)\pi z/2L]sen\,[(2j-1)\pi/2]\,,
\end{aligned}$$

em que usamos,

$$\int_0^{+L}\cos(k\pi x/2L)dx = \frac{2L}{k\pi}sen\,(k\pi/2)\,.$$

Obtemos assim outra série para a constante 1,

$$1 = \frac{4}{\pi}\sum_{j=1}\frac{1}{2j-1}\cos[(2j-1)\pi z/2L]sen\,[(2j-1)\pi/2]\,. \tag{27}$$

A figura 3 mostra a série acima para alguns termos da série.

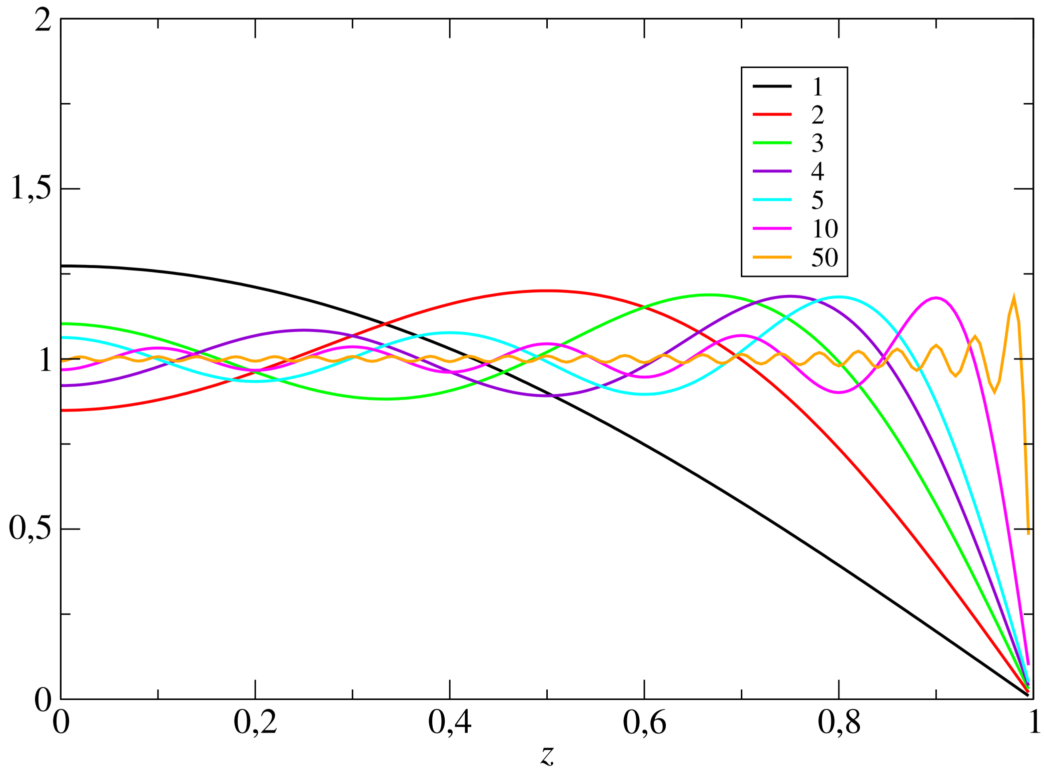

Fig. 3. Série representando a constante 1 no intervalo $0 \leq z \leq L$*, para* $L = 1$.

A solução u_1 *é então,*

$$u_1(\rho, z) = \frac{2}{L} \sum_{j=1} \frac{I_0((2j-1)\pi\rho/2L)}{I_0((2j-1)\pi a/2L)} \cos[(2j-1)\pi z/2L] \times \times \int_0^{+L} f(x) \cos[(2j-1)\pi x/2L] dx \,.$$

Podemos escrever a solução acima como,

$$u_1(\rho, z) = \int_0^{+L} G(x, \rho, z) f(x) dx \,,$$

com,

$$G(x, \rho, z) = \frac{2}{L} \sum_{j=1} \frac{I_0((2j-1)\pi\rho/2L)}{I_0((2j-1)\pi a/2L)} \times \times \cos[(2j-1)\pi z/2L] \cos[(2j-1)\pi x/2L] \,.$$

Se $f(z) = f_0$ constante temos,

$$\begin{aligned} u_1(\rho, z) &= \frac{2}{L}\sum_{j=1}\frac{I_0((2j-1)\pi\rho/2L)}{I_0((2j-1)\pi a/2L)}\cos[(2j-1)\pi z/2L]\times \\ &\quad \times\int_0^{+L} f(x)\cos[(2j-1)\pi x/2L]dx\,, \\ &= \frac{2}{L}\sum_{j=1}\frac{I_0((2j-1)\pi\rho/2L)}{I_0((2j-1)\pi a/2L)}\cos[(2j-1)\pi z/2L]\times \\ &\quad \times f_0\int_0^{+L}\cos[(2j-1)\pi x/2L]dx\,, \\ &= \frac{4f_0}{\pi}\sum_{j=1}\frac{I_0((2j-1)\pi\rho/2L)}{I_0((2j-1)\pi a/2L)}\cos[(2j-1)\pi z/2L]\times \\ &\quad \times\frac{1}{2j-1}sen\,[(2j-1)\pi/2]\,. \end{aligned}$$

Se $f_0 = 0$ temos $u_1 = 0$.

(b) Cálculo de u_2.

Escrevemos u_2 como,

$$\begin{aligned} u_2(\rho, z) &= c_0 z + d_0 \\ &\quad + \sum_{\lambda>0} J_0(\lambda\rho)a_{2\lambda}senh\,\lambda(L-z)\,. \end{aligned}$$

As condições de contorno para u_2 são,

$$\begin{aligned} u_2(a, z) &= 0 = c_0 z + d_0 + \sum_{\lambda>0} J_0(\lambda a)a_{2\lambda}senh\,\lambda(L-z)\,, \\ \frac{\partial u_2(\rho,0)}{\partial z} &= g(\rho) = c_0 - \sum_{\lambda>0}\lambda J_0(\lambda\rho)a_{2\lambda}cosh\,\lambda L\,, \\ u_2(\rho, L) &= 0 = c_0 L + d_0\,. \end{aligned}$$

Satisfazemos as condições acima escolhendo,

$$\begin{aligned} &c_0 = d_0 = 0\,, \\ &J_0(\lambda_j a) = 0\,, \quad j = 1, 2, \ldots \end{aligned}$$

A condição para $g(\rho)$ é uma série em funções de Bessel,

$$g(\rho) = -\sum_{j=1} \lambda_j J_0(\lambda_j \rho) a_{2j} \cosh \lambda_j L \,,$$

logo,

$$-\lambda_j a_{2j} \cosh \lambda_j L = \frac{2}{a^2 J_1^2(\lambda_j a)} \int_0^a \rho J_0(\lambda_j \rho) g(\rho) d\rho \,,$$

ou,

$$a_{2j} = -\frac{2}{\lambda_j a^2 J_1^2(\lambda_j a) \cosh \lambda_j L} \int_0^a \rho J_0(\lambda_j \rho) g(\rho) d\rho \,.$$

Explicitamente,

$$g(\rho) = \frac{2}{a^2} \sum_{j=1} \frac{J_0(\lambda_j \rho)}{J_1^2(\lambda_j a)} \int_0^a x J_0(\lambda_j x) g(x) dx \,.$$

Se $g(\rho) = g_0$ constante,

$$\begin{aligned}
g(\rho) &= \frac{2}{a^2} \sum_{j=1} \frac{J_0(\lambda_j \rho)}{J_1^2(\lambda_j a)} \int_0^a x J_0(\lambda_j x) g(x) dx \,, \\
g_0 &= \frac{2}{a^2} \sum_{j=1} \frac{J_0(\lambda_j \rho)}{J_1^2(\lambda_j a)} g_0 \int_0^a x J_0(\lambda_j x) dx \,, \\
g_0 &= \frac{2 g_0}{a} \sum_{j=1} \frac{J_0(\lambda_j \rho)}{\lambda_j J_1(\lambda_j a)} \,, \\
g_0 &= g_0 \,.
\end{aligned}$$

A solução u_2 é então,

$$u_2(\rho, z) = -\frac{2}{a^2} \sum_{j=1} \frac{J_0(\lambda_j \rho) \operatorname{senh} \lambda_j (L - z)}{\lambda_j J_1^2(\lambda_j a) \cosh \lambda_j L} \int_0^a x J_0(\lambda_j x) g(x) dx \,.$$

Podemos escrever a solução acima como,

$$u_2(\rho, z) = \int_0^a G(x, \rho, z) g(x) dx \,,$$

com,

$$G(x, \rho, z) = -\frac{2}{a^2} \sum_{j=1} \frac{J_0(\lambda_j \rho) \operatorname{senh} \lambda_j (L - z)}{\lambda_j J_1^2(\lambda_j a) \cosh \lambda_j L} x J_0(\lambda_j x) \,.$$

Se $g(\rho) = g_0$ *constante,*

$$\begin{aligned} u_2(\rho, z) &= -\frac{2}{a^2} \sum_{j=1} \frac{J_0(\lambda_j \rho) senh\, \lambda_j (L - z)}{\lambda_j J_1^2(\lambda_j a) cosh\, \lambda_j L} \int_0^a x J_0(\lambda_j x) g(x) dx\,, \\ &= -\frac{2g_0}{a} \sum_{j=1} \frac{J_0(\lambda_j \rho) senh\, \lambda_j (L - z)}{\lambda_j^2 J_1(\lambda_j a) cosh\, \lambda_j L}\,. \end{aligned}$$

Se $g_0 = 0$ *temos* $u_2 = 0$.

(c) Cálculo de u_3.

Escrevemos u_3 *como,*

$$\begin{aligned} u_3(\rho, z) &= c_0 z + d_0 \\ &+ \sum_{\lambda > 0} J_0(\lambda \rho) a_{2\lambda} cosh\, \lambda z\,. \end{aligned}$$

As condições de contorno nos dão as equações,

$$\begin{aligned} u_3(a, z) &= 0 = c_0 z + d_0 \\ &+ \sum_{\lambda > 0} J_0(\lambda a) a_{2\lambda} cosh\, \lambda z\,, \\ \frac{\partial u_3(\rho, 0)}{\partial z} &= 0 = c_0\,, \\ u_3(\rho, L) &= h(\rho) = c_0 L + d_0 \\ &+ \sum_{\lambda > 0} J_0(\lambda \rho) a_{2\lambda} cosh\, \lambda L\,. \end{aligned}$$

Satisfazemos as condições acima escolhendo,

$$\begin{aligned} & c_0 = d_0 = 0\,, \\ & J_0(\lambda_j a) = 0\,, \quad j = 1, 2, \ldots \end{aligned}$$

A equação para $h(\rho)$ *é assim uma série de funções de Bessel,*

$$h(\rho) = \sum_{j=1} J_0(\lambda_j \rho) a_{2j} cosh\, \lambda_j L\,,$$

logo,

$$a_{2j}\, cosh\, \lambda_j L = \frac{2}{a^2 J_1^2(\lambda_j a)} \int_0^a \rho J_0(\lambda_j \rho) h(\rho) d\rho\,,$$

ou,

$$a_{2j} = \frac{2}{a^2 J_1^2(\lambda_j a)\, cosh\, \lambda_j L} \int_0^a \rho J_0(\lambda_j \rho) h(\rho) d\rho\,.$$

Temos então,

$$h(\rho) = \frac{2}{a^2} \sum_{j=1} \frac{J_0(\lambda_j \rho)}{J_1^2(\lambda_j a)} \int_0^a x J_0(\lambda_j x) h(x) dx\,.$$

A solução u_3 é assim,

$$\begin{aligned} u_3(\rho, z) &= c_0 z + d_0 + \sum_{j=1} J_0(\lambda_j \rho) a_{2j}\, cosh\, \lambda_j z\,, \\ &= \frac{2}{a^2} \sum_{j=1} \frac{J_0(\lambda_j \rho)\, cosh\, \lambda_j z}{J_1^2(\lambda_j a)\, cosh\, \lambda_j L} \int_0^a x J_0(\lambda_j x) h(x) dx\,. \end{aligned}$$

Podemos escrever a solução acima como,

$$u_3(\rho, z) = \int_0^a G(x, \rho, z) h(x) dx\,,$$

com,

$$G(x, \rho, z) = \frac{2}{a^2} \sum_{j=1} \frac{J_0(\lambda_j \rho)\, cosh\, \lambda_j z}{J_1^2(\lambda_j a)\, cosh\, \lambda_j L} x J_0(\lambda_j x)\,.$$

Se $h(\rho) = h_0$ constante,

$$\begin{aligned} u_3(\rho, z) &= \frac{2}{a^2} \sum_{j=1} \frac{J_0(\lambda_j \rho)\, cosh\, \lambda_j z}{J_1^2(\lambda_j a)\, cosh\, \lambda_j L} \int_0^a x J_0(\lambda_j x) h(x) dx\,, \\ &= \frac{2}{a^2} \sum_{j=1} \frac{J_0(\lambda_j \rho)\, cosh\, \lambda_j z}{J_1^2(\lambda_j a)\, cosh\, \lambda_j L} h_0 \int_0^a x J_0(\lambda_j x) dx\,, \\ &= \frac{2h_0}{a} \sum_{j=1} \frac{J_0(\lambda_j \rho)\, cosh\, \lambda_j z}{\lambda_j J_1(\lambda_j a)\, cosh\, \lambda_j L}\,. \end{aligned}$$

Se $h_0 = 0$ temos $u_3 = 0$.

(d) Se as funções nas condições de contorno são constantes,

$$
\begin{aligned}
u(\rho,z) &= \frac{4f_0}{\pi}\sum_{j=1}\frac{I_0((2j-1)\pi\rho/2L)}{I_0((2j-1)\pi a/2L)}\cos[(2j-1)\pi z/2L]\times \\
&\times\frac{1}{2j-1}sen\,[(2j-1)\pi/2] \\
&-\frac{2g_0}{a}\sum_{j=1}\frac{J_0(\lambda_j\rho)senh\,\lambda_j(L-z)}{\lambda_j^2J_1(\lambda_j a)cosh\,\lambda_j L} \\
&+\frac{2h_0}{a}\sum_{j=1}\frac{J_0(\lambda_j\rho)cosh\,\lambda_j z}{\lambda_j J_1(\lambda_j a)cosh\,\lambda_j L}\,.
\end{aligned}
$$

4. Considerando o intervalo $0 \le \rho \le a$, $0 \le z \le L$, encontre a solução $u(\rho,z)$ com as condições de contorno,

$$
\begin{aligned}
&u(a,z) = f(z)\,, \\
&u(\rho,0) = g(\rho)\,, \\
&\frac{\partial u(\rho,L)}{\partial z} = h(\rho)\,.
\end{aligned}
$$

Escrevemos a solução como uma soma de três funções, cada uma satisfazendo uma das condições de contorno e se anulando nas outras,

$$u(\rho,z) = u_1(\rho,z) + u_2(\rho,z) + u_3(\rho,z)\,,$$

com,

$$
\begin{aligned}
&u_1(a,z) = f(z)\,,\ \ u_1(\rho,0) = 0\,,\ \ \frac{\partial u_1(\rho,L)}{\partial z} = 0\,, \\
&u_2(a,z) = 0\,,\ \ u_2(\rho,0) = g(\rho)\,,\ \ \frac{\partial u_2(\rho,L)}{\partial z} = 0\,, \\
&u_3(a,z) = 0\,,\ \ u_3(\rho,0) = 0\,,\ \ \frac{\partial u_3(\rho,L)}{\partial z} = h(\rho)\,.
\end{aligned}
$$

(a) Cálculo de u_1.
Escrevemos u_1 na forma,

$$
\begin{aligned}
u_1(\rho,z) &= c_0 z + d_0 \\
&+\sum_{\lambda>0} I_0(\lambda\rho)[a_{1\lambda}\cos\lambda z + a_{2\lambda}sen\,\lambda z]\,.
\end{aligned}
$$

As condições de contorno são,

$$\begin{aligned}
u_1(a,z) &= f(z) = c_0 z + d_0 \\
&+ \sum_{\lambda>0} I_0(\lambda a)[a_{1\lambda}\cos\lambda z + a_{2\lambda} sen\,\lambda z]\,, \\
u_1(\rho,0) &= 0 = d_0 + \sum_{\lambda>0} I_0(\lambda\rho) a_{1\lambda}\,, \\
\frac{\partial u_1(\rho,L)}{\partial z} &= 0 = c_0 + \sum_{\lambda>0} I_0(\lambda\rho)[-\lambda a_{1\lambda} sen\lambda L + \lambda a_{2\lambda}\cos\lambda L]\,.
\end{aligned}$$

Satisfazemos as relações acima escolhendo,

$$\begin{aligned}
&c_0 = d_0 = 0\,, \\
&a_{1\lambda} = 0\,, \\
&\lambda_j L = (2j-1)\frac{\pi}{2}\,.
\end{aligned}$$

A equação para $f(z)$ é então,

$$f(z) = \sum_{j=1} I_0(\lambda_j a) a_{2j} sen\,[(2j-1)\pi z/2L]\,,$$

logo,

$$I_0(\lambda_j a) a_{2j} = \frac{2}{L}\int_0^L f(x) sen\,((2j-1)\pi x/2L)\,dx\,,$$

ou,

$$a_{2j} = \frac{2}{L I_0(\lambda_j a)}\int_0^L f(x) sen\,((2j-1)\pi x/2L)\,dx\,.$$

Portanto,

$$f(z) = \sum_{j=1} sen\,[(2j-1)\pi z/2L]\frac{2}{L}\int_0^L f(x) sen\,((2j-1)\pi x/2L)\,dx\,.$$

A equação acima nos dá outra expressão para a função delta em $0 \leq x \leq L$,

$$\delta(z-x) = \frac{2}{L}\sum_{j=1} sen\,[(2j-1)\pi z/2L] sen\,((2j-1)\pi x/2L)\,.$$

Se $f(z) = f_0$ *constante,*

$$
\begin{aligned}
f(z) &= \sum_{j=1} sen\,[(2j-1)\pi z/2L]\frac{2}{L}\int_0^L f(x)sen\,((2j-1)\pi x/2L)\,dx\,,\\
f_0 &= \sum_{j=1} sen\,[(2j-1)\pi z/2L]\frac{2}{L}f_0\int_0^L sen\,((2j-1)\pi x/2L)\,,\\
f_0 &= \frac{4f_0}{\pi}\sum_{j=1}\frac{sen\,[(2j-1)\pi z/2L]}{2j-1}\,,\\
f_0 &= f_0\,,
\end{aligned}
$$

em que usamos,

$$\int_0^L sen\,((2j-1)\pi x/2L)\,dx = \frac{2L}{(2j-1)\pi}\,,$$

e,

$$1 = \frac{4}{\pi}\sum_{j=1}\frac{sen\,[(2j-1)\pi z/2L]}{2j-1}\,. \tag{28}$$

A solução u_1 *é então,*

$$
\begin{aligned}
u_1(\rho,z) &= \sum_{\lambda>0} I_0(\lambda\rho)a_{2\lambda}sen\,\lambda z\,,\\
&= \frac{2}{L}\sum_{j=1}\frac{I_0(\lambda_j\rho)}{I_0(\lambda_j a)}sen\,[(2j-1)\pi z/2L]\int_0^L f(x)sen\,((2j-1)\pi x/2L)\,dx\,.
\end{aligned}
$$

Podemos escrever a solução acima na forma,

$$u_1(\rho,z) = \int_0^L G(x,\rho,z)f(x)dx\,,$$

com,

$$G(x,\rho,z) = \frac{2}{L}\sum_{j=1}\frac{I_0(\lambda_j\rho)}{I_0(\lambda_j a)}sen\,[(2j-1)\pi z/2L]sen\,((2j-1)\pi x/2L)\,.$$

Se $f(z) = f_0$ *constante,*

$$
\begin{aligned}
u_1(\rho,z) &= \frac{2}{L}\sum_{j=1}\frac{I_0(\lambda_j\rho)}{I_0(\lambda_j a)} sen\,[(2j-1)\pi z/2L]\int_0^L f(x) sen\,((2j-1)\pi x/2L)\,dx\,,\\
&= \frac{2}{L}\sum_{j=1}\frac{I_0(\lambda_j\rho)}{I_0(\lambda_j a)} sen\,[(2j-1)\pi z/2L] f_0\int_0^L sen\,((2j-1)\pi x/2L)\,dx\,,\\
&= \frac{4f_0}{\pi}\sum_{j=1}\frac{I_0(\lambda_j\rho)}{I_0(\lambda_j a)}\frac{sen\,[(2j-1)\pi z/2L]}{2j-1}\,.
\end{aligned}
$$

Se $f_0 = 0$ temos $u_1 = 0$.

(b) Cálculo de u_2.

A solução u_2 pode ser escrita como,

$$
u_2(\rho,z) = c_0 z + d_0 + \sum_{\lambda>0} J_0(\lambda\rho) a_{2\lambda} cosh\,\lambda(L-z)\,.
$$

As condições de contorno ficam,

$$
\begin{aligned}
&u_2(a,z) = 0 = c_0 z + d_0 + \sum_{\lambda>0} J_0(\lambda a) a_{2\lambda} cosh\,\lambda(L-z)\,,\\
&u_2(\rho,0) = g(\rho) = d_0 + \sum_{\lambda>0} J_0(\lambda\rho) a_{2\lambda} cosh\,\lambda L\,,\\
&\frac{\partial u_2(\rho,L)}{\partial z} = 0 = c_0\,.
\end{aligned}
$$

Satisfazemos as condições acima escolhendo,

$$
\begin{aligned}
&c_0 = d_0 = 0\,,\\
&J_0(\lambda_j a) = 0\,,\quad j = 1,2,\ldots
\end{aligned}
$$

A expansão para $g(\rho)$ é então,

$$
g(\rho) = \sum_{j=1} J_0(\lambda_j\rho) a_{2j} cosh\,\lambda_j L\,,
$$

logo,

$$
a_{2j} cosh\,\lambda_j L = \frac{2}{a^2 J_1^2(\lambda_j a)}\int_0^a \rho J_0(\lambda_j\rho) g(\rho) d\rho\,,
$$

ou,

$$a_{2j} = \frac{2}{a^2 J_1^2(\lambda_j a) cosh\, \lambda_j L} \int_0^a \rho J_0(\lambda_j \rho) g(\rho) d\rho \,.$$

A série para $g(\rho)$ *é assim,*

$$g(\rho) = \frac{2}{a^2} \sum_{j=1} \frac{J_0(\lambda_j \rho)}{J_1^2(\lambda_j a)} \int_0^a x J_0(\lambda_j x) g(x) dx \,.$$

Temos então,

$$u_2(\rho, z) \;=\; \frac{2}{a^2} \sum_{j=1} \frac{J_0(\lambda_j \rho) cosh\, \lambda_j (L - z)}{J_1^2(\lambda_j a) cosh\, \lambda_j L} \int_0^a x J_0(\lambda_j x) g(x) dx \,.$$

Podemos escrever a solução acima na forma,

$$u_2(\rho, z) = \int_0^a G(x, \rho, z) g(x) dx \,,$$

com,

$$G(x, \rho, z) = \frac{2}{a^2} \sum_{j=1} \frac{J_0(\lambda_j \rho) cosh\, \lambda_j (L - z)}{J_1^2(\lambda_j a) cosh\, \lambda_j L} x J_0(\lambda_j x) \,.$$

Se $g(\rho) = g_0$ *constante,*

$$\begin{aligned} u_2(\rho, z) &= \frac{2}{a^2} \sum_{j=1} \frac{J_0(\lambda_j \rho) cosh\, \lambda_j (L - z)}{J_1^2(\lambda_j a) cosh\, \lambda_j L} \int_0^a x J_0(\lambda_j x) g(x) dx \,, \\ &= \frac{2}{a^2} \sum_{j=1} \frac{J_0(\lambda_j \rho) cosh\, \lambda_j (L - z)}{J_1^2(\lambda_j a) cosh\, \lambda_j L} g_0 \int_0^a x J_0(\lambda_j x) dx \,, \\ &= \frac{2 g_0}{a} \sum_{j=1} \frac{J_0(\lambda_j \rho) cosh\, \lambda_j (L - z)}{\lambda_j J_1(\lambda_j a) cosh\, \lambda_j L} \,. \end{aligned}$$

Se $g_0 = 0$ *temos* $u_2 = 0$.

(c) Cálculo de u_3.

Escrevemos u_3 *como,*

$$u_3(\rho, z) \;=\; c_0 z + d_0 + \sum_{\lambda > 0} J_0(\lambda \rho) a_{2\lambda} senh\, \lambda z \,.$$

As condições de contorno ficam,

$$u_3(a,z) = 0 = c_0 z + d_0 + \sum_{\lambda>0} J_0(\lambda a) a_{2\lambda} senh\,\lambda z\,,$$
$$u_3(\rho,0) = 0 = d_0\,,$$
$$\frac{\partial u_3(\rho,L)}{\partial z} = h(\rho) = c_0 L + d_0 + \sum_{\lambda>0} J_0(\lambda\rho) a_{2\lambda}\lambda cosh\,\lambda L\,.$$

Podemos satisfazer as condições acima escolhendo,

$$c_0 = d_0 = 0\,,$$
$$J_0(\lambda_j a) = 0\,,\quad j = 1,2,\ldots$$

Temos uma expansão de $h(\rho)$ em funções de Bessel,

$$h(\rho) = \sum_{j=1} J_0(\lambda_j\rho) a_{2j}\lambda_j cosh\,\lambda_j L\,,$$

logo,

$$\lambda_j a_{2j} cosh\lambda_j L = \frac{2}{a^2 J_1^2(\lambda_j a)} \int_0^a \rho J_0(\lambda_j\rho) h(\rho) d\rho\,,$$

ou,

$$a_{2j} = \frac{2}{a^2\lambda_j J_1^2(\lambda_j a) cosh\,\lambda_j L} \int_0^a \rho J_0(\lambda_j\rho) h(\rho) d\rho\,.$$

A série para $h(\rho)$ é assim,

$$h(\rho) = \frac{2}{a^2} \sum_{j=1} \frac{J_0(\lambda_j\rho)}{J_1^2(\lambda_j a)} \int_0^a x J_0(\lambda_j x) h(x) dx\,.$$

Se $h(\rho) = h_0$ constante,

$$\begin{aligned}
h(\rho) &= \frac{2}{a^2} \sum_{j=1} \frac{J_0(\lambda_j\rho)}{J_1^2(\lambda_j a)} \int_0^a x J_0(\lambda_j x) h(x) dx\,,\\
h_0 &= \frac{2}{a^2} \sum_{j=1} \frac{J_0(\lambda_j\rho)}{J_1^2(\lambda_j a)} h_0 \int_0^a x J_0(\lambda_j x) dx\,,\\
h_0 &= \frac{2h_0}{a} \sum_{j=1} \frac{J_0(\lambda_j\rho)}{\lambda_j J_1(\lambda_j a)}\,,\\
h_0 &= h_0\,,
\end{aligned}$$

como esperado.

A solução u_3 é então,

$$u_3(\rho,z)=\frac{2}{a^2}\sum_{j=1}\frac{J_0(\lambda_j\rho)senh\,\lambda_j z}{\lambda_j J_1^2(\lambda_j a)cosh\,\lambda_j L}\int_0^a xJ_0(\lambda_j x)h(x)dx\,.$$

Podemos escrever a solução acima na forma,

$$u_3(\rho,z)=\int_0^a G(x,\rho,z)h(x)dx\,,$$

com,

$$G(x,\rho,z)=\frac{2}{a^2}\sum_{j=1}\frac{J_0(\lambda_j\rho)senh\,\lambda_j z}{\lambda_j J_1^2(\lambda_j a)cosh\,\lambda_j L}xJ_0(\lambda_j x\,.$$

Se $h(\rho)=h_0$ constante temos,

$$\begin{aligned}u_3(\rho,z) &= \frac{2}{a^2}\sum_{j=1}\frac{J_0(\lambda_j\rho)senh\,\lambda_j z}{\lambda_j J_1^2(\lambda_j a)cosh\,\lambda_j L}\int_0^a xJ_0(\lambda_j x)h(x)dx\,,\\ &= \frac{2}{a^2}\sum_{j=1}\frac{J_0(\lambda_j\rho)senh\,\lambda_j z}{\lambda_j J_1^2(\lambda_j a)cosh\,\lambda_j L}h_0\int_0^a xJ_0(\lambda_j x)dx\,,\\ &= \frac{2h_0}{a}\sum_{j=1}\frac{J_0(\lambda_j\rho)senh\,\lambda_j z}{\lambda_j^2 J_1(\lambda_j a)cosh\,\lambda_j L}\,,\end{aligned}$$

Se $h_0=0$ temos $u_3=0$.

(d) Se as funções das condições de contorno são constantes temos,

$$\begin{aligned}u(\rho,z) &= \frac{4f_0}{\pi}\sum_{j=1}\frac{I_0(\lambda_j\rho)}{I_0(\lambda_j a)}\frac{sen\,[(2j-1)\pi z/2L]}{2j-1}\\ &\quad+\frac{2g_0}{a}\sum_{j=1}\frac{J_0(\lambda_j\rho)cosh\,\lambda_j(L-z)}{\lambda_j J_1(\lambda_j a)cosh\,\lambda_j L}\\ &\quad+\frac{2h_0}{a}\sum_{j=1}\frac{J_0(\lambda_j\rho)senh\,\lambda_j z}{\lambda_j^2 J_1(\lambda_j a)cosh\,\lambda_j L}\,.\end{aligned}$$

5. Considere um cilindro infinito de raio a. Calcule a solução $u(\rho,z)$ da equação de Laplace com a condição de contorno $u(a,z)=f(z)$ ([7], probl. 6.104).

Escrevemos a solução como,

$$u(\rho, z) = \sum_\lambda I_0(\lambda\rho)[a_{1\lambda}\cos\lambda z + a_{2\lambda} sen\,\lambda z]\,,$$

e a condição de contorno fica,

$$f(z) = \sum_\lambda I_0(\lambda a)[a_{1\lambda}\cos\lambda z + a_{2\lambda} sen\,\lambda z]\,.$$

Como temos agora $-\infty < z < +\infty$*, a expansão de* $f(z)$ *é uma integral de Fourier,*

$$\begin{aligned} f(z) &= \int_0^\infty [A(\lambda)\cos\lambda z + B(\lambda) sen\,\lambda z] d\lambda\,,\\ A(\lambda) &= \frac{1}{\pi}\int_{-\infty}^{+\infty} f(v)\cos\lambda v dv\,,\\ B(\lambda) &= \frac{1}{\pi}\int_{-\infty}^{+\infty} f(v) sen\,\lambda v dv\,. \end{aligned}$$

Precisamos então escrever a solução e a condição de contorno como integrais em lugar de somas,

$$\begin{aligned} u(\rho, z) &= \int_0^\infty I_0(\lambda\rho)[a_1(\lambda)\cos\lambda z + a_2(\lambda) sen\,\lambda z] d\lambda\,,\\ f(z) &= \int_0^\infty I_0(\lambda a)[a_1(\lambda)\cos\lambda z + a_2(\lambda) sen\,\lambda z] d\lambda\,. \end{aligned}$$

Temos então,

$$\begin{aligned} A(\lambda) = I_0(\lambda a) a_1(\lambda) \Rightarrow a_1(\lambda) = \frac{A(\lambda)}{I_0(\lambda a)}\,,\\ B(\lambda) = I_0(\lambda a) a_2(\lambda) \Rightarrow a_2(\lambda) = \frac{B(\lambda)}{I_0(\lambda a)}\,, \end{aligned}$$

e a solução fica,

$$\begin{aligned} u(\rho, z) \;=\; & \int_0^\infty \frac{I_0(\lambda\rho)}{I_0(\lambda a)}\cos\lambda z d\lambda \frac{1}{\pi}\int_{-\infty}^{+\infty} f(v)\cos\lambda v dv\\ & + \int_0^\infty \frac{I_0(\lambda\rho)}{I_0(\lambda a)} sen\,\lambda z d\lambda \frac{1}{\pi}\int_{-\infty}^{+\infty} f(v) sen\,\lambda v dv\,, \end{aligned}$$

ou, usando $\cos(a-b) = \cos a \cos b + sen\, a sen\, b$,

$$u(\rho,z) \;=\; \frac{1}{\pi}\int_0^{\infty} d\lambda \int_{-\infty}^{+\infty} dv \frac{I_0(\lambda\rho)}{I_0(\lambda a)} f(v)\cos\lambda(z-v)\,.$$

Podemos escrever a solução acima na forma,

$$u(\rho,z) = \int_{-\infty}^{+\infty} G(v,\rho,z) f(v) dv\,,$$

com,

$$G(v,\rho,z) = \frac{1}{\pi}\int_0^{\infty} d\lambda \frac{I_0(\lambda\rho)}{I_0(\lambda a)}\cos\lambda(z-v)\,.$$

Se $f(z) = f_0$ *constante em* $-L \leq z \leq +L$,

$$\begin{aligned} u(\rho,z) &= \frac{1}{\pi}\int_0^{\infty} d\lambda \int_{-\infty}^{+\infty} dv \frac{I_0(\lambda\rho)}{I_0(\lambda a)} f(v)\cos\lambda(z-v)\,, \\ &= \frac{f_0}{\pi}\int_0^{\infty} d\lambda \frac{I_0(\lambda\rho)}{I_0(\lambda a)} \int_{-L}^{+L} dv \cos\lambda(z-v)\,, \\ &= \frac{f_0}{\pi}\int_0^{\infty} d\lambda \frac{I_0(\lambda\rho)}{I_0(\lambda a)} \frac{1}{\lambda}[-sen\,\lambda(z-L) + sen\,\lambda(z+L)]\,. \end{aligned}$$

Se $f_0 = 0$ *temos* $u = 0$.

6. Considere um cilindro infinito de raio a. Encontre a solução $u(\rho,z)$ com a condição de contorno,

$$\frac{\partial u(a,z)}{\partial \rho} = f(z)\,.$$

7. Considerando o intervalo $a \leq \rho \leq b$, $0 \leq z \leq L$, encontre a solução $u(\rho,z)$ com as condições de contorno,

$$\begin{aligned} u(a,z) &= f(z)\,, \\ u(b,z) &= g(z)\,, \\ u(\rho,0) &= h(\rho)\,, \\ u(\rho,L) &= v(\rho)\,. \end{aligned}$$

Escrevemos a solução como uma soma de quatro funções, cada uma satisfazendo uma das condições de contorno e se anulando nas outras, como antes,

$$u(\rho, z) = u_1(\rho, z) + u_2(\rho, z) + u_3(\rho, z) + u_4(\rho, z)\,,$$

com,

$$\begin{aligned}
&u_1(a, z) = f(z)\,, \;\; u_1(b, z) = 0\,, \;\; u_1(\rho, 0) = 0\,, \;\; u_1(\rho, L) = 0\,,\\
&u_2(a, z) = 0\,, \;\; u_2(b, z) = g(z)\,, \;\; u_2(\rho, 0) = 0\,, \;\; u_2(\rho, L) = 0\,,\\
&u_3(a, z) = 0\,, \;\; u_3(b, z) = 0\,, \;\; u_3(\rho, 0) = h(\rho)\,, \;\; u_3(\rho, L) = 0\,,\\
&u_4(a, z) = 0\,, \;\; u_4(b, z) = 0\,, \;\; u_4(\rho, 0) = 0\,, \;\; u_4(\rho, L) = v(\rho)\,.
\end{aligned}$$

(a) Cálculo de u_1.
Como não temos agora o ponto $\rho = 0$, escrevemos u_1 na forma,

$$\begin{aligned}
u_1(\rho, z) &= (c_0 z + d_0)(a_0 + b_0 \ln \rho)\\
&\quad + \sum_{\lambda > 0} [b_{1\lambda} I_0(\lambda\rho) + b_{2\lambda} K_0(\lambda\rho)][a_{1\lambda} \cos \lambda z + a_{2\lambda} sen\, \lambda z]\,.
\end{aligned}$$

As condições de contorno são,

$$\begin{aligned}
u_1(a, z) &= f(z) = (c_0 z + d_0)(a_0 + b_0 \ln a)\\
&\quad + \sum_{\lambda > 0} [b_{1\lambda} I_0(\lambda a) + b_{2\lambda} K_0(\lambda a)][a_{1\lambda} \cos \lambda z + a_{2\lambda} sen\, \lambda z]\,,\\
u_1(b, z) &= 0 = (c_0 z + d_0)(a_0 + b_0 \ln b)\\
&\quad + \sum_{\lambda > 0} [b_{1\lambda} I_0(\lambda b) + b_{2\lambda} K_0(\lambda b)][a_{1\lambda} \cos \lambda z + a_{2\lambda} sen\, \lambda z]\,,\\
u_1(\rho, 0) &= 0 = d_0(a_0 + b_0 \ln \rho)\\
&\quad + \sum_{\lambda > 0} [b_{1\lambda} I_0(\lambda\rho) + b_{2\lambda} K_0(\lambda\rho)] a_{1\lambda}\,,\\
u_1(\rho, L) &= 0 = (c_0 L + d_0)(a_0 + b_0 \ln \rho)\\
&\quad + \sum_{\lambda > 0} [b_{1\lambda} I_0(\lambda\rho) + b_{2\lambda} K_0(\lambda\rho)][a_{1\lambda} \cos \lambda L + a_{2\lambda} sen\, \lambda L]\,.
\end{aligned}$$

Podemos satisfazer as equações acima escolhendo,

$$\begin{aligned}
&c_0 = d_0 = a_0 = b_0 = a_{1\lambda} = 0\,,\\
&\lambda_j L = \pi j\,, \;\; j = 1, 2, \ldots\\
&b_{1j} I_0(\lambda_j b) + b_{2j} K_0(\lambda_j b) = 0\,.
\end{aligned}$$

A última equação acima determina a relação entre b_{1j} *e* b_{2j}*,*

$$b_{2j} = -b_{1j}\frac{I_0(\lambda_j b)}{K_0(\lambda_j b)}\,.$$

A condição para $f(z)$ *nos dá uma série de Fourier de senos,*

$$f(z) = \sum_{j=1}[b_{1j}I_0(\lambda_j a) + b_{2j}K_0(\lambda_j a)]a_{2j}sen\,\lambda_j z\,,$$

ou, substituindo b_{2j}*,*

$$f(z) = \sum_{j=1}[I_0(\lambda_j a)K_0(\lambda_j b) - I_0(\lambda_j b)K_0(\lambda_j a)]\frac{b_{1j}a_{2j}}{K_0(\lambda_j b)}sen\,\lambda_j z\,.$$

Definindo,

$$V_n(\lambda_j\rho) \equiv I_n(\lambda_j\rho)K_n(\lambda_j b) - I_n(\lambda_j b)K_n(\lambda_j\rho)\,, \qquad (29)$$

podemos escrever,

$$f(z) = \sum_{j=1}\frac{b_{1j}a_{2j}V_0(\lambda_j a)}{K_0(\lambda_j b)}sen\,\lambda_j z\,.$$

Portanto,

$$\frac{b_{1j}a_{2j}V_0(\lambda_j a)}{K_0(\lambda_j b)} = \frac{2}{L}\int_0^L f(z)sen\,(j\pi z/L)\,dz\,, \quad j = 0, 1, 2, \ldots$$

ou,

$$b_{1j}a_{2j} = \frac{K_0(\lambda_j b)}{V_0(\lambda_j a)}\frac{2}{L}\int_0^L f(z)sen\,(j\pi z/L)\,dz\,, \quad j = 0, 1, 2, \ldots$$

A expansão de $f(z)$ *é assim,*

$$f(z) = \frac{2}{L}\sum_{j=1}sen\,(j\pi z/L)\int_0^L f(x)sen\,(j\pi x/L)\,dx\,.$$

Se $f(z) = f_0$ *constante,*

$$
\begin{aligned}
f(z) &= \frac{2}{L}\sum_{j=1} sen\,(j\pi z/L)\int_0^L f(x) sen\,(j\pi x/L)\,dx\,, \\
f_0 &= f_0\frac{2}{L}\sum_{j=1} sen\,(j\pi z/L)\int_0^L sen\,(j\pi x/L)\,dx\,, \\
f_0 &= f_0\frac{4}{\pi}\sum_{j=1}\frac{sen\,[(2j-1)\pi z/L]}{2j-1}\,, \\
f_0 &= f_0\,,
\end{aligned}
$$

como esperado.

A solução u_1 é então ($\lambda_j = j\pi/L$),

$$
u_1(\rho,z) = \frac{2}{L}\sum_{j=1}\frac{V_0(\lambda_j\rho)}{V_0(\lambda_j a)} sen\,(j\pi z/L)\int_0^L f(x) sen\,(j\pi x/L)\,dx\,.
$$

Podemos escrever a solução acima como,

$$
u_1(\rho,z) = \int_0^L G(x,\rho,z) f(x) dx\,,
$$

com,

$$
G(x,\rho,z) = \frac{2}{L}\sum_{j=1}\frac{V_0(\lambda_j\rho)}{V_0(\lambda_j a)} sen\,(j\pi z/L) sen\,(j\pi x/L)\,.
$$

Se $f(z) = f_0$ constante,

$$
\begin{aligned}
u_1(\rho,z) &= \sum_{j=1}\frac{V_0(\lambda_j\rho)}{V_0(\lambda_j a)} sen\,\lambda_j z\frac{2}{L}\int_0^L f(x) sen\,(j\pi x/L)\,dx\,, \\
&= \sum_{j=1}\frac{V_0(\lambda_j\rho)}{V_0(\lambda_j a)} sen\,\lambda_j z\frac{2}{L} f_0\int_0^L sen\,(j\pi x/L)\,dx\,, \\
&= \sum_{j=1}\frac{V_0(\pi(2j-1)\rho/L)}{V_0(\pi(2j-1)a/L)} sen\,[\pi(2j-1)z/L]\frac{2}{L} f_0\frac{2L}{(2j-1)\pi}\,, \\
&= \frac{4f_0}{\pi}\sum_{j=1}\frac{V_0(\pi(2j-1)\rho/L)}{V_0(\pi(2j-1)a/L)}\frac{sen\,[\pi(2j-1)z/L]}{2j-1}\,.
\end{aligned}
$$

Se $f_0 = 0$ temos $u_1 = 0$.

(b) Cálculo de u_2.

Escrevemos u_2 na forma,

$$\begin{aligned} u_2(\rho, z) &= (c_0 z + d_0)(a_0 + b_0 \ln \rho) \\ &\quad + \sum_{\lambda>0} [b_{1\lambda} I_0(\lambda\rho) + b_{2\lambda} K_0(\lambda\rho)][a_{1\lambda} \cos \lambda z + a_{2\lambda} sen\, \lambda z]\,. \end{aligned}$$

As condições de contorno são,

$$\begin{aligned} u_2(a, z) &= 0 = (c_0 z + d_0)(a_0 + b_0 \ln a) \\ &\quad + \sum_{\lambda>0} [b_{1\lambda} I_0(\lambda a) + b_{2\lambda} K_0(\lambda a)][a_{1\lambda} \cos \lambda z + a_{2\lambda} sen\, \lambda z]\,, \\ u_2(b, z) &= g(z) = (c_0 z + d_0)(a_0 + b_0 \ln b) \\ &\quad + \sum_{\lambda>0} [b_{1\lambda} I_0(\lambda b) + b_{2\lambda} K_0(\lambda b)][a_{1\lambda} \cos \lambda z + a_{2\lambda} sen\, \lambda z]\,, \\ u_2(\rho, 0) &= 0 = d_0(a_0 + b_0 \ln \rho) \\ &\quad + \sum_{\lambda>0} [b_{1\lambda} I_0(\lambda\rho) + b_{2\lambda} K_0(\lambda\rho)] a_{1\lambda}\,, \\ u_2(\rho, L) &= 0 = (c_0 L + d_0)(a_0 + b_0 \ln \rho) \\ &\quad + \sum_{\lambda>0} [b_{1\lambda} I_0(\lambda\rho) + b_{2\lambda} K_0(\lambda\rho)][a_{1\lambda} \cos \lambda L + a_{2\lambda} sen\, \lambda L]\,. \end{aligned}$$

Podemos satisfazer as equações acima escolhendo,

$$\begin{aligned} & c_0 = d_0 = a_0 = b_0 = a_{1\lambda} = 0\,, \\ & \lambda_j L = \pi j\,, \quad j = 1, 2, \ldots \\ & b_{1j} I_0(\lambda_j a) + b_{2j} K_0(\lambda_j a) = 0\,. \end{aligned}$$

A última equação acima determina a relação entre b_{1j} e b_{2j},

$$b_{2j} = -b_{1j} \frac{I_0(\lambda_j a)}{K_0(\lambda_j a)}\,.$$

Temos uma expansão de $g(z)$ em série de Fourier de senos,

$$g(z) = \sum_{j=1} [b_{1j} I_0(\lambda_j b) + b_{2j} K_0(\lambda_j b)] a_{2j} sen\, \lambda_j z\,,$$

ou,

$$g(z) = \sum_{j=1} [I_0(\lambda_j b) K_0(\lambda_j a) - I_0(\lambda_j a) K_0(\lambda_j b)] \frac{b_{1j} a_{2j}}{K_0(\lambda_j a)} sen\, \lambda_j z \, .$$

Definindo,

$$W_n(\lambda_j \rho) \equiv I_n(\lambda_j \rho) K_n(\lambda_j a) - I_n(\lambda_j a) K_n(\lambda_j \rho) \, , \tag{30}$$

temos,

$$g(z) = \sum_{j=1} \frac{b_{1j} a_{2j} W_0(\lambda_j b)}{K_0(\lambda_j a)} sen\, \lambda_j z \, .$$

Temos uma série de Fourier de senos, logo,

$$\frac{b_{1j} a_{2j} W_0(\lambda_j b)}{K_0(\lambda_j a)} = \frac{2}{L} \int_0^L g(z) sen\, (j\pi z/L) \, dz \, , \quad j = 0, 1, 2, \ldots$$

A série para $g(z)$ é assim,

$$g(z) = \sum_{j=1} sen\, (j\pi z/L) \frac{2}{L} \int_0^L g(z) sen\, (j\pi z/L) \, dz \, .$$

A solução u_2 é então ($\lambda_j = j\pi/L$),

$$u_2(\rho, z) \;=\; \sum_{j=1} \frac{W_0(\lambda_j \rho)}{W_0(\lambda_j b)} sen\, \lambda_j z \frac{2}{L} \int_0^L g(x) sen\, (j\pi x/L) \, dx \, .$$

Podemos escrever a solução acima na forma,

$$u_2(\rho, z) = \int_0^L G(x, \rho, z) g(x) dx \, ,$$

com,

$$G(x, \rho, z) = \sum_{j=1} \frac{W_0(\lambda_j \rho)}{W_0(\lambda_j b)} sen\, \lambda_j z \frac{2}{L} sen\, (j\pi x/L) \, .$$

Se $g(z) = g_0$ constante,

$$
\begin{aligned}
u_2(\rho,z) &= \sum_{j=1} \frac{W_0(\lambda_j\rho)}{W_0(\lambda_j b)} sen\,\lambda_j z \frac{2}{L}\int_0^L g(x) sen\,(j\pi x/L)\,dx\,,\\
&= \sum_{j=1} \frac{W_0(\lambda_j\rho)}{W_0(\lambda_j b)} sen\,\lambda_j z \frac{2}{L} g_0 \int_0^L sen\,(j\pi x/L)\,dx\,,\\
&= \frac{4g_0}{\pi}\sum_{j=1} \frac{W_0((2j-1)\pi\rho/L)}{W_0((2j-1)\pi b/L)} \frac{sen\,[(2j-1)\pi z/L]}{2j-1}\,.
\end{aligned}
$$

Se $g_0 = 0$ temos $u_2 = 0$.

(c) Cálculo de u_3.

Escrevemos u_3 como,

$$
\begin{aligned}
u_3(\rho,z) &= (c_0 z + d_0)(a_0 + b_0 \ln\rho)\\
&\quad + \sum_{\lambda>0} [b_{1\lambda} J_0(\lambda\rho) + b_{2\lambda} Y_0(\lambda\rho)] senh\lambda(L-z)\,,
\end{aligned}
$$

e as condições de contorno são,

$$
\begin{aligned}
u_3(a,z) &= 0 = (c_0 z + d_0)(a_0 + b_0 \ln a)\\
&\quad + \sum_{\lambda>0} [b_{1\lambda} J_0(\lambda a) + b_{2\lambda} Y_0(\lambda a)] senh\,\lambda(L-z)\,,\\
u_3(b,z) &= 0 = (c_0 z + d_0)(a_0 + b_0 \ln b)\\
&\quad + \sum_{\lambda>0} [b_{1\lambda} J_0(\lambda b) + b_{2\lambda} Y_0(\lambda b)] senh\,\lambda(L-z)\,,\\
u_3(\rho,0) &= h(\rho) = d_0(a_0 + b_0 \ln\rho)\\
&\quad + \sum_{\lambda>0} [b_{1\lambda} J_0(\lambda\rho) + b_{2\lambda} Y_0(\lambda\rho)] senh\lambda L\,,\\
u_3(\rho,L) &= 0 = (c_0 L + d_0)(a_0 + b_0 \ln\rho)\,.
\end{aligned}
$$

Satisfazemos as equações acima escolhendo,

$$
\begin{aligned}
&c_0 = d_0 = a_0 = b_0 = 0\,,\\
&b_{1j} J_0(\lambda_j a) + b_{2j} Y_0(\lambda_j a) = 0\,,\\
&b_{1j} J_0(\lambda_j b) + b_{2j} Y_0(\lambda_j b) = 0\,.
\end{aligned}
$$

Portanto, λ_j é determinado por,

$$J_0(\lambda_j a)Y_0(\lambda_j b) - J_0(\lambda_j b)Y_0(\lambda_j a) = 0\,,$$

ou,

$$U_0(\lambda_j b) = 0\,, \tag{31}$$

com,

$$U_n(\lambda_j \rho) \equiv J_n(\lambda_j \rho)Y_n(\lambda_j a) - J_n(\lambda_j a)Y_n(\lambda_j \rho)\,. \tag{32}$$

Também temos,

$$b_{2j} = -b_{1j}\frac{J_0(\lambda_j a)}{Y_0(\lambda_j a)} = -b_{1j}\frac{J_0(\lambda_j b)}{Y_0(\lambda_j b)}\,.$$

Temos assim uma expansão de $h(\rho)$ *em funções de Bessel,*

$$h(\rho) = \sum_j \frac{b_{1j}\,senh\lambda_j L}{Y_0(\lambda_j a)} U_0(\lambda_j \rho)\,.$$

portanto,

$$\frac{b_{1j}\,senh\lambda_j L}{Y_0(\lambda_j a)} = \frac{\displaystyle\int_a^b \rho h(\rho) U_0(\lambda_j \rho) d\rho}{\displaystyle\int_a^b \rho [U_0(\lambda_j \rho)]^2 d\rho}\,,$$

ou,

$$b_{1j} = \frac{Y_0(\lambda_j a)}{senh\lambda_j L}\frac{\displaystyle\int_a^b \rho h(\rho) U_0(\lambda_j \rho) d\rho}{\displaystyle\int_a^b \rho [U_0(\lambda_j \rho)]^2 d\rho}\,.$$

A expansão de $h(\rho)$ *é assim,*

$$h(\rho) = \sum_j U_0(\lambda_j \rho)\frac{\displaystyle\int_a^b x h(x) U_0(\lambda_j x) dx}{\displaystyle\int_a^b y [U_0(\lambda_j y)]^2 dy}\,.$$

Se $h(\rho) = h_0$ *constante,*

$$
\begin{aligned}
h(\rho) &= \sum_j U_0(\lambda_j\rho)\frac{\int_a^b xh(x)U_0(\lambda_j x)dx}{\int_a^b y[U_0(\lambda_j y)]^2 dy}\,, \\
h_0 &= \sum_j U_0(\lambda_j\rho)\frac{h_0\int_a^b xU_0(\lambda_j x)dx}{\int_a^b y[U_0(\lambda_j y)]^2 dy}\,.
\end{aligned}
$$

Das equações acima obtemos duas novas relações,

$$
\delta(\rho - x) = \sum_j \frac{U_0(\lambda_j\rho)}{\int_a^b y[U_0(\lambda_j y)]^2 dy} xU_0(\lambda_j x)\,, \quad a \leq \rho \leq b\,, \tag{33}
$$

$$
1 = \sum_j U_0(\lambda_j\rho)\frac{\int_a^b xU_0(\lambda_j x)dx}{\int_a^b y[U_0(\lambda_j y)]^2 dy}\,. \tag{34}
$$

É interessante fazer alguns gráficos. A figura 4 mostra o gráfico de $U_0(\lambda b)$ *como função de* λ*, para* $a = 1$ *e* $b = 2$*. Desse gráfico podemos obter algumas raízes da equação (31),*

$$
\begin{aligned}
\lambda_j \;=\;& 3,128,\ 6,275,\ 9,42,\ 12,562,\ 15,705,\ 18,847,\ 21,988,\ 25,131, \\
& 28,272,\ 31,414,\ 34,556,\ 37,697,\ 40,840,\ 43,981,\ 47,123, \ldots
\end{aligned}
$$

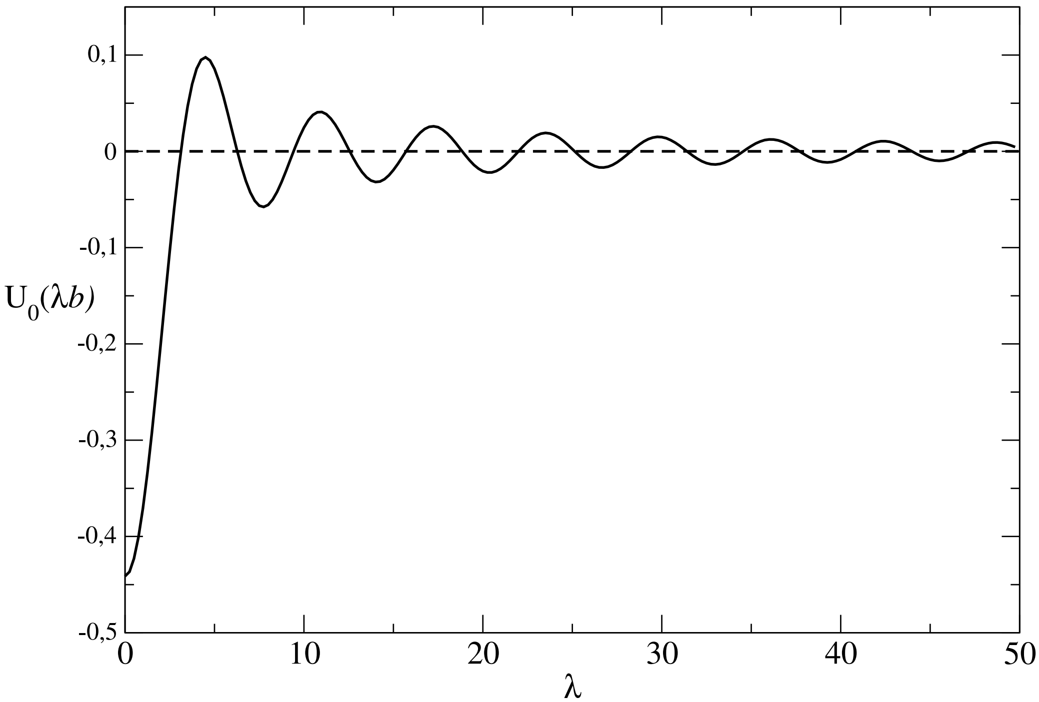

Fig. 4. Gráfico de $U_0(\lambda b)$ como função de λ, para $a = 1$ e $b = 2$.

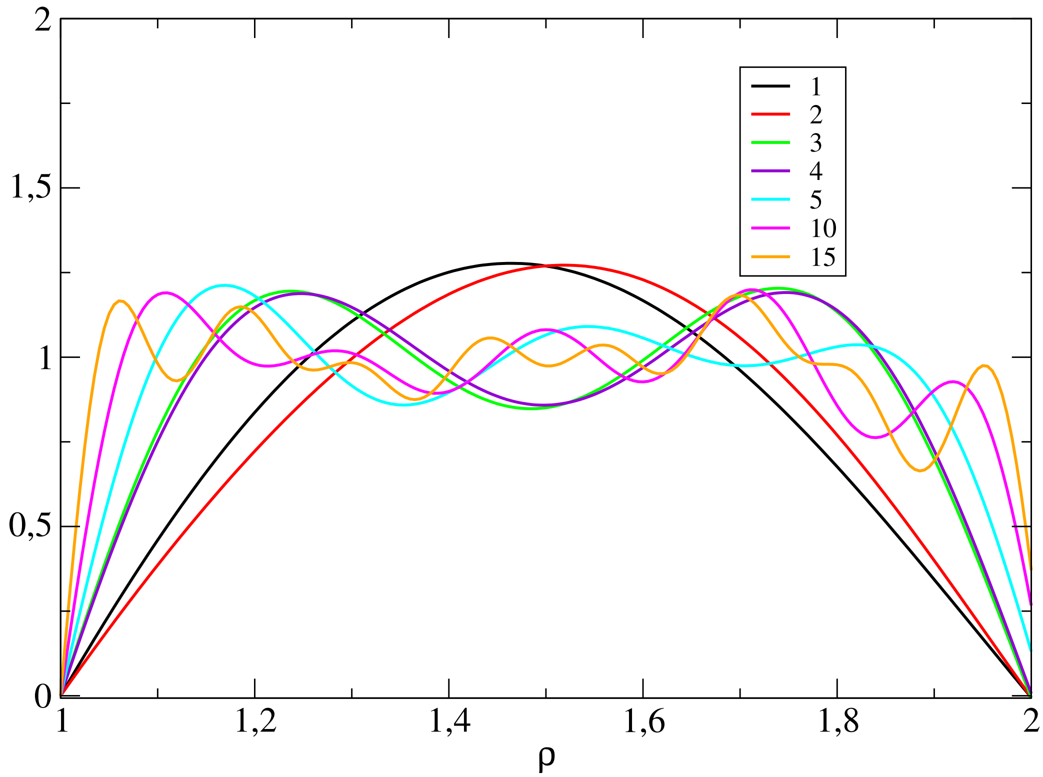

Fig. 5. Gráfico da série 33, representando a constante 1 em $a \leq \rho \leq b$, para alguns termos da soma, com $a = 1$ e $b = 2$.

A solução u_3 é então,

$$u_3(\rho, z) = \sum_{j=1} U_0(\lambda_j \rho) \frac{senh\lambda_j(L-z)}{senh\lambda_j L} \frac{\int_a^b xh(x)U_0(\lambda_j x)dx}{\int_a^b y[U_0(\lambda_j y)]^2 dy} .$$

Podemos escrever a solução acima na forma,

$$u_3(\rho, z) = \int_a^b G(x, \rho, z)h(x)dx ,$$

com,

$$G(x, \rho, z) = \sum_{j=1} U_0(\lambda_j \rho) \frac{senh\lambda_j(L-z)}{senh\lambda_j L} \frac{xU_0(\lambda_j x)}{\int_a^b y[U_0(\lambda_j y)]^2 dy} .$$

Se $h(\rho) = h_0$ constante temos,

$$\begin{aligned} u_3(\rho, z) &= \sum_{j=1} U_0(\lambda_j \rho) \frac{senh\lambda_j(L-z)}{senh\lambda_j L} \frac{\int_a^b xh(x)U_0(\lambda_j x)dx}{\int_a^b y[U_0(\lambda_j y)]^2 dy} , \\ &= h_0 \sum_{j=1} U_0(\lambda_j \rho) \frac{senh\lambda_j(L-z)}{senh\lambda_j L} \frac{\int_a^b xU_0(\lambda_j x)dx}{\int_a^b y[U_0(\lambda_j y)]^2 dy} . \end{aligned}$$

Se $h_0 = 0$ temos $u_3 = 0$.

(d) Cálculo de u_4.

Escrevemos u_4 como,

$$\begin{aligned} u_4(\rho, z) &= (c_0 z + d_0)(a_0 + b_0 \ln \rho) \\ &\quad + \sum_{\lambda > 0} [b_{1\lambda} J_0(\lambda \rho) + b_{2\lambda} Y_0(\lambda \rho)][a_{1\lambda} cosh\lambda z + a_{2\lambda} senh\lambda z] , \end{aligned}$$

e as condições de contorno são,

$$\begin{aligned}
u_4(a,z) &= 0 = (c_0 z + d_0)(a_0 + b_0 \ln a) \\
&+ \sum_{\lambda>0} [b_{1\lambda} J_0(\lambda a) + b_{2\lambda} Y_0(\lambda a)][a_{1\lambda} cosh \lambda z + a_{2\lambda} senh\, \lambda z]\,, \\
u_4(b,z) &= 0 = (c_0 z + d_0)(a_0 + b_0 \ln b) \\
&+ \sum_{\lambda>0} [b_{1\lambda} J_0(\lambda b) + b_{2\lambda} Y_0(\lambda b)][a_{1\lambda} cosh \lambda z + a_{2\lambda} senh\, \lambda z]\,, \\
u_4(\rho,0) &= 0 = d_0(a_0 + b_0 \ln \rho) \\
&+ \sum_{\lambda>0} [b_{1\lambda} J_0(\lambda \rho) + b_{2\lambda} Y_0(\lambda \rho)] a_{1\lambda}\,, \\
u_4(\rho,L) &= v(\rho) = (c_0 L + d_0)(a_0 + b_0 \ln \rho) \\
&+ \sum_{\lambda>0} [b_{1\lambda} J_0(\lambda \rho) + b_{2\lambda} Y_0(\lambda \rho)][a_{1\lambda} cosh \lambda L + a_{2\lambda} senh\, \lambda L]\,.
\end{aligned}$$

Satisfazemos as equações acima escolhendo,

$$\begin{aligned}
c_0 = d_0 = a_0 = b_0 = a_{1\lambda} = 0\,, \\
b_{1j} J_0(\lambda_j a) + b_{2j} Y_0(\lambda_j a) = 0\,, \\
b_{1j} J_0(\lambda_j b) + b_{2j} Y_0(\lambda_j b) = 0\,.
\end{aligned}$$

Portanto, λ_j *é determinado por,*

$$J_0(\lambda_j a) Y_0(\lambda_j b) - J_0(\lambda_j b) Y_0(\lambda_j a) = 0\,,$$

ou,

$$U_0(\lambda_j b) = 0\,.$$

e,

$$b_{2j} = -b_{1j} \frac{J_0(\lambda_j a)}{Y_0(\lambda_j a)} = -b_{1j} \frac{J_0(\lambda_j b)}{Y_0(\lambda_j b)}\,.$$

Temos assim uma expansão de $v(\rho)$ *em funções de Bessel,*

$$v(\rho) = \sum_{j=1} U_0(\lambda_j \rho) \frac{b_{1j} a_{2j}}{Y_0(\lambda_j a)} senh\, \lambda_j L\,,$$

com,

$$\frac{b_{1j}a_{2j}}{Y_0(\lambda_j a)} senh\,\lambda_j L = \frac{\displaystyle\int_a^b xv(x)U_0(\lambda_j x)dx}{\displaystyle\int_a^b y[U_0(\lambda_j y)]^2 dy} \,.$$

Obtemos então,

$$u_4(\rho, z) \;=\; \sum_{j=1} \frac{senh\lambda_j z}{senh\,\lambda_j L} U_0(\lambda_j \rho) \frac{\displaystyle\int_a^b xv(x)U_0(\lambda_j x)dx}{\displaystyle\int_a^b y[U_0(\lambda_j y)]^2 dy} \,, \quad U_0(\lambda_j b) = 0 \,.$$

Podemos escrever a solução acima na forma,

$$u_4(\rho, z) = \int_a^b G(x, \rho, z)v(x)dx \,,$$

com,

$$G(x, \rho, z) = \sum_{j=1} \frac{senh\lambda_j z}{senh\,\lambda_j L} U_0(\lambda_j \rho) \frac{xU_0(\lambda_j x)}{\displaystyle\int_a^b y[U_0(\lambda_j y)]^2 dy} \,.$$

Se $v(\rho) = v_0$ *constante,*

$$\begin{aligned} u_4(\rho, z) \;&=\; \sum_{j=1} \frac{senh\lambda_j z}{senh\,\lambda_j L} U_0(\lambda_j \rho) \frac{\displaystyle\int_a^b xv(x)U_0(\lambda_j x)dx}{\displaystyle\int_a^b y[U_0(\lambda_j y)]^2 dy} \,, \\ &=\; v_0 \sum_{j=1} \frac{senh\lambda_j z}{senh\,\lambda_j L} U_0(\lambda_j \rho) \frac{\displaystyle\int_a^b xU_0(\lambda_j x)dx}{\displaystyle\int_a^b y[U_0(\lambda_j y)]^2 dy} \,. \end{aligned}$$

Se $v_0 = 0$ *temos* $u_4 = 0$.

(e) Se as funções das condições de contorno são constantes a solução é,

$$
\begin{aligned}
u(\rho, z) \quad = \quad & \frac{4f_0}{\pi}\sum_{j=1} \frac{V_0(\pi(2j-1)\rho/L)}{V_0(\pi(2j-1)a/L)} \frac{sen\,[(2j-1)\pi z/L]}{2j-1} \\
& + \frac{4g_0}{\pi}\sum_{j=1} \frac{W_0((2j-1)\pi\rho/L)}{W_0((2j-1)\pi b/L)} \frac{sen\,[(2j-1)\pi z/L]}{2j-1} \\
& + h_0 \sum_{j=1} U_0(\lambda_j \rho) \frac{senh\lambda_j (L-z)}{senh\lambda_j L} \frac{\displaystyle\int_a^b xU_0(\lambda_j x)dx}{\displaystyle\int_a^b y[U_0(\lambda_j y)]^2 dy} \\
& + v_0 \sum_{j=1} \frac{senh\lambda_j z}{senh\,\lambda_j L} U_0(\lambda_j \rho) \frac{\displaystyle\int_a^b xU_0(\lambda_j x)dx}{\displaystyle\int_a^b y[U_0(\lambda_j y)]^2 dy}\,.
\end{aligned}
$$

8. Considere uma casca cilíndrica infinita de raio interno a e raio interno b. Encontre a solução $u(\rho, z)$ com as condições de contorno,

$$
\begin{aligned}
u(a, z) = f(z)\,, \\
u(b, z) = g(z)\,.
\end{aligned}
$$

9. Encontre a solução $u(\rho, z)$ para as equações:

(a)

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right) + \frac{\partial^2 u}{\partial z^2} = -f(\rho, z)\,;$$

(b)

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{du}{d\rho}\right) + \frac{\partial^2 u}{\partial z^2} = +\alpha^2\,;$$

com α constante,

(c)

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{du}{d\rho}\right) + \frac{\partial^2 u}{\partial z^2} = -\alpha^2\,;$$

com α constante,

(d)

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{du}{d\rho}\right) + \frac{\partial^2 u}{\partial z^2} = +\alpha^2 u\,;$$

com α constante,

(e)

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{du}{d\rho}\right)+\frac{\partial^2 u}{\partial z^2}=-\alpha^2 u\,;$$

com α constante e,

$$\begin{aligned}&0\le\rho\le a\,,\;\;0\le z\le L\,,\\&u(a,z)=\mu(z)\,,\;\;u(\rho,0)=\nu_1(\rho)\,,\\&u(\rho,L)=\nu_2(\rho)\,.\end{aligned}$$

10. Encontre a solução $u(\rho)$ para as equações:

(a)

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{du}{d\rho}\right)+\frac{\partial^2 u}{\partial z^2}=-f(\rho,z)\,;$$

(b)

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{du}{d\rho}\right)+\frac{\partial^2 u}{\partial z^2}=+\alpha^2\,;$$

com α constante,

(c)

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{du}{d\rho}\right)+\frac{\partial^2 u}{\partial z^2}=-\alpha^2\,;$$

com α constante,

(d)

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{du}{d\rho}\right)+\frac{\partial^2 u}{\partial z^2}=+\alpha^2 u\,;$$

com α constante,

(e)

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{du}{d\rho}\right)+\frac{\partial^2 u}{\partial z^2}=-\alpha^2 u\,;$$

com α constante e,

$$\begin{aligned}&a\le\rho\le b\,,\;\;0\le z\le L\,,\\&u(a,z)=\mu_1(z)\,,\;\;u(b,z)=\mu_2(z)\,,\\&u(\rho,0)=\nu_1(\rho)\,,\;\;u(\rho,L)=\nu_2(\rho)\,.\end{aligned}$$

5 Considerando $u = u(\rho, \varphi)$

Substituindo $u(\rho, \varphi) = R(\rho)\Phi(\varphi)$ em (1), vem,

$$\frac{\rho}{R}(\rho R')' + \frac{\Phi''}{\Phi} = 0\,.$$

Da equação acima temos,

$$\frac{\rho}{R}(\rho R')' = -\frac{\Phi''}{\Phi} \equiv +\lambda^2\,. \tag{35}$$

Da relação acima temos,

$$\Phi'' + \lambda^2\Phi = 0\,, \tag{36}$$

$$\rho^2 R'' + \rho R' - \lambda^2 R = 0\,. \tag{37}$$

A equação (36) possui solução,

$$\Phi(\varphi) = b_1 \cos\lambda\varphi + b_2 \mathrm{sen}\,\lambda\varphi\,. \tag{38}$$

A equação (37) é a *equação diferencial de Cauchy ou Euler*, com solução,

$$R(\rho) = \frac{a_1}{\rho^\lambda} + a_2\rho^\lambda\,. \tag{39}$$

Notemos que as soluções acima são para $\lambda \neq 0$. Se $\lambda = 0$ as funções R e Φ são constantes. Se $\lambda = 0$, temos de (35),

$$\Phi'' = 0\,, \tag{40}$$

$$(\rho R')' = 0\,. \tag{41}$$

As soluções das equações acima são,

$$\Phi(\varphi) = c_0\varphi + d_0\,, \tag{42}$$

$$R(r) = a_0 + b_0 \ln\rho\,. \tag{43}$$

Escolhemos $c_0 =$ para termos Φ unívoca. Pelo princípio da superposição, a solução geral é uma soma das possíveis soluções individuais,

$$u(\rho, \varphi) = a_0 + b_0 \ln\rho + \sum_{\lambda>0} \left(\frac{a_{1\lambda}}{\rho^\lambda} + a_{2\lambda}\rho^\lambda\right)[b_{1\lambda}\cos\lambda\varphi + b_{2\lambda}\mathrm{sen}\,\lambda\varphi]\,. \tag{44}$$

A determinação das constantes e dos possíveis valores de λ depende das condições de contorno.

6 Problemas

1. Considerando o intervalo $0 \leq \rho \leq a$, $0 \leq \varphi \leq 2\pi$, encontre a solução $u(\rho, \varphi)$ com a condição de contorno,

$$u(a, \varphi) = f(\varphi)\,.$$

A solução geral, com λ inteiro, para que a solução seja unicamente definida, e finita em $\rho = 0$, é

$$u(\rho, \varphi) = a_0 + \sum_{j=1} \rho^j [b_{1j} \cos j\varphi + b_{2j} sen\, j\varphi]\,.$$

Usando a condição de contorno temos,

$$f(\varphi) = a_0 + \sum_{j=1} a^j [b_{1j} \cos j\varphi + b_{2j} sen\, j\varphi]\,.$$

A expressão acima é uma série de Fourier, logo,

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx\,, \\ a^j b_{1j} &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos jx\, dx\,, \\ a^j b_{2j} &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) sen\, jx\, dx\,, \quad j = 1, 2, \ldots \end{aligned}$$

o que completa a solução do problema,

$$\begin{aligned} u(\rho, \varphi) \;=\; & \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx \\ & + \sum_{j=1} \frac{\rho^j}{a^j} \cos\, j\varphi \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos jx\, dx \\ & + \sum_{j=1} \frac{\rho^j}{a^j} sen\, j\varphi \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) sen\, jx\, dx\,. \end{aligned}$$

Se $f(\varphi) = f_0$ constante,

$$u(\rho, \varphi) = f_0\,.$$

2. Considerando o intervalo $0 \leq \rho \leq a$, $0 \leq \varphi \leq 2\pi$, encontre a solução $u(\rho, \varphi)$ com a condição de contorno,

$$\frac{\partial u(a, \varphi)}{\partial \rho} = f(\varphi)\,.$$

A solução geral é, como antes,

$$u(\rho, \varphi) = a_0 + \sum_{j=1} \rho^j [b_{1j} \cos\, j\varphi + b_{2j} sen\, j\varphi]\,.$$

Usando a condição de contorno temos,

$$\frac{\partial u(a, \varphi)}{\partial \rho} = f(\varphi) = \sum_{j=1} j a^{j-1} [b_{1j} \cos j\varphi + b_{2j} sen\, j\varphi]\,.$$

A expressão acima é uma série de Fourier, logo,

$$\begin{aligned} j a^{j-1} b_{1j} &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos jx\, dx\,, \\ j a^{j-1} b_{2j} &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) sen\, jx\, dx\,, \quad j = 1, 2, \ldots, \end{aligned}$$

ou,

$$\begin{aligned} b_{1j} &= \frac{1}{\pi j a^{j-1}} \int_{-\pi}^{\pi} f(x) \cos jx\, dx\,, \\ b_{2j} &= \frac{1}{\pi j a^{j-1}} \int_{-\pi}^{\pi} f(x) sen\, jx\, dx\,, \quad j = 1, 2, \ldots. \end{aligned}$$

A solução do problema é então,

$$\begin{aligned} u(\rho, \varphi) &= a_0 + \sum_{j=1} \frac{\rho^j}{\pi j a^{j-1}} \cos\, j\varphi \int_{-\pi}^{\pi} f(x) \cos jx\, dx \\ &\quad + \sum_{j=1} \frac{\rho^j}{\pi j a^{j-1}} sen\, j\varphi \int_{-\pi}^{\pi} f(x) sen\, jx\, dx\,, \end{aligned}$$

com a_0 indeterminado. Se $f(\varphi) = f_0$ constante,

$$u(\rho, \varphi) = a_0\,.$$

Como $\partial u/\partial\rho = 0$, vemos que $f_0 = 0$. Portanto não podemos ter $\partial u/\partial\rho(a)$ constante $\neq 0$.

3. Considerando o intervalo $a \leq \rho \leq b$, $0 \leq \varphi \leq 2\pi$, encontre a solução $u(\rho,\varphi)$ com as condições de contorno,

$$\begin{aligned} u(a,\varphi) &= f(\varphi)\,, \\ u(b,\varphi) &= g(\varphi)\,. \end{aligned}$$

Escrevemos a solução como,

$$u(\rho,\varphi) = u_1(\rho,\varphi) + u_2(\rho,\varphi)\,,$$

com,

$$\begin{aligned} &u_1(a,\varphi) = f(\varphi)\,,\ \ u_1(b,\varphi) = 0\,, \\ &u_2(a,\varphi) = 0\,,\ \ u_2(b,\varphi) = g(\varphi)\,. \end{aligned}$$

(a) Cálculo de $u_1(\rho,\varphi)$.
Escrevemos,

$$u_1(\rho,\varphi) = a_0 + b_0 \ln\rho + \sum_{\lambda>0}\left(\frac{a_{1\lambda}}{\rho^\lambda} + a_{2\lambda}\rho^\lambda\right)[b_{1\lambda}\cos\lambda\varphi + b_{2\lambda} sen\,\lambda\varphi]\,.$$

e as condições de contorno,

$$\begin{aligned} u_1(a,\varphi) &= f(\varphi) = a_0 + b_0\ln a + \sum_\lambda\left(\frac{a_{1\lambda}}{a^\lambda} + a_{2\lambda}a^\lambda\right)[b_{1\lambda}\cos\lambda\varphi + b_{2\lambda} sen\,\lambda\varphi]\,, \\ u_1(b,\varphi) &= 0 = a_0 + b_0\ln b + \sum_\lambda\left(\frac{a_{1\lambda}}{b^\lambda} + a_{2\lambda}b^\lambda\right)[b_{1\lambda}\cos\lambda\varphi + b_{2\lambda} sen\,\lambda\varphi]\,. \end{aligned}$$

Podemos satisfazer as condições acima escolhendo,

$$\begin{aligned} &a_0 + b_0\ln b = 0\,, \\ &\frac{a_{1\lambda}}{b^\lambda} + a_{2\lambda}b^\lambda = 0 \Rightarrow a_{2\lambda} = -\frac{a_{1\lambda}}{b^{2\lambda}}\,, \end{aligned}$$

logo, fazendo $a_{1\lambda} = 1$,

$$u_1(\rho,\varphi) = a_0 + b_0 \ln \rho + \sum_{\lambda} \left(\frac{1}{\rho^{\lambda}} - \frac{\rho^{\lambda}}{b^{2\lambda}} \right) [b_{1\lambda} \cos \lambda\varphi + b_{2\lambda} sen\, \lambda\varphi] \, ,$$

a condição de contorno para $f(\varphi)$ fica,

$$f(\varphi) = a_0 + b_0 \ln a + \sum_{\lambda} \left(\frac{1}{a^{\lambda}} - \frac{a^{\lambda}}{b^{2\lambda}} \right) [b_{1\lambda} \cos \lambda\varphi + b_{2\lambda} sen\, \lambda\varphi] \, .$$

Temos assim uma série de Fourier para $f(\varphi)$, portanto,

$$\begin{aligned} a_0 + b_0 \ln a &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx \, , \\ \left(\frac{1}{a^{\lambda}} - \frac{a^{\lambda}}{b^{2\lambda}} \right) b_{1\lambda} &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos \lambda x \, dx \, , \\ \left(\frac{1}{a^{\lambda}} - \frac{a^{\lambda}}{b^{2\lambda}} \right) b_{2\lambda} &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) sen\, \lambda x \, dx \, , \end{aligned}$$

ou,

$$\begin{aligned} b_{1\lambda} &= \frac{\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos \lambda x \, dx}{\left(\frac{1}{a^{\lambda}} - \frac{a^{\lambda}}{b^{2\lambda}} \right)} \, , \\ b_{2\lambda} &= \frac{\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) sen\, \lambda x \, dx}{\left(\frac{1}{a^{\lambda}} - \frac{a^{\lambda}}{b^{2\lambda}} \right)} \, . \end{aligned}$$

Com isso a solução u_1 fica,

$$\begin{aligned} u_1(\rho,\varphi) &= a_0 + b_0 \ln \rho \\ &\quad + \sum_{j=1} \left(\frac{1}{\rho^{j}} - \frac{\rho^{j}}{b^{2j}} \right) \cos j\varphi \frac{\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos jx \, dx}{\left(\frac{1}{a^{j}} - \frac{a^{j}}{b^{2j}} \right)} \\ &\quad + \sum_{j=1} \left(\frac{1}{\rho^{j}} - \frac{\rho^{j}}{b^{2j}} \right) sen\, j\varphi \frac{\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) sen\, jx \, dx}{\left(\frac{1}{a^{j}} - \frac{a^{j}}{b^{2j}} \right)} \, , \end{aligned}$$

com,

$$a_0 = \frac{\ln b}{\ln b/a} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx\,,$$
$$b_0 = \frac{\dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} f(x)\, dx}{\ln a/b}\,.$$

Se $f(\varphi) = f_0$ *constante,*

$$u_1(\rho, \varphi) = a_0 + b_0 \ln \rho\,,$$

com,

$$a_0 = \frac{\ln b}{\ln b/a} f_0\,,$$
$$b_0 = \frac{f_0}{\ln a/b}\,.$$

(b) Cálculo de $u_2(\rho, \varphi)$.
Escrevemos, como antes,

$$u_2(\rho, \varphi) = a_0 + b_0 \ln \rho + \sum_{\lambda > 0} \left(\frac{a_{1\lambda}}{\rho^{\lambda}} + a_{2\lambda} \rho^{\lambda} \right) [b_{1\lambda} \cos \lambda\varphi + b_{2\lambda} sen\, \lambda\varphi]\,.$$

e as condições de contorno,

$$u_2(a, \varphi) = a_0 + b_0 \ln a + \sum_{\lambda} \left(\frac{a_{1\lambda}}{a^{\lambda}} + a_{2\lambda} a^{\lambda} \right) [b_{1\lambda} \cos \lambda\varphi + b_{2\lambda} sen\, \lambda\varphi]\,,$$
$$u_2(b, \varphi) = g(\varphi) = a_0 + b_0 \ln b + \sum_{\lambda} \left(\frac{a_{1\lambda}}{b^{\lambda}} + a_{2\lambda} b^{\lambda} \right) [b_{1\lambda} \cos \lambda\varphi + b_{2\lambda} sen\, \lambda\varphi]\,.$$

Temos agora,

$$a_0 + b_0 \ln a = 0\,,$$
$$\frac{a_{1\lambda}}{a^{\lambda}} + a_{2\lambda} a^{\lambda} = 0 \Rightarrow a_{2\lambda} = -\frac{a_{1\lambda}}{a^{2\lambda}}\,,$$

logo, fazendo $a_{1\lambda} = 1$,

$$u_2(\rho,\varphi) = a_0 + b_0 \ln \rho + \sum_{\lambda} \left(\frac{1}{\rho^\lambda} - \frac{\rho^\lambda}{a^{2\lambda}} \right) \left[b_{1\lambda} \cos \lambda\varphi + b_{2\lambda} sen\, \lambda\varphi \right],$$

a condição de contorno para $g(\varphi)$ fica,

$$g(\varphi) = a_0 + b_0 \ln b + \sum_{\lambda} \left(\frac{1}{b^\lambda} - \frac{b^\lambda}{a^{2\lambda}} \right) \left[b_{1\lambda} \cos \lambda\varphi + b_{2\lambda} sen\, \lambda\varphi \right].$$

Temos assim uma série de Fourier para $g(\varphi)$, portanto,

$$a_0 + b_0 \ln b = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x)\, dx,$$
$$\left(\frac{1}{b^\lambda} - \frac{b^\lambda}{a^{2\lambda}} \right) b_{1\lambda} = \frac{1}{\pi} \int_{-\pi}^{\pi} g(x) \cos \lambda x\, dx,$$
$$\left(\frac{1}{b^\lambda} - \frac{b^\lambda}{a^{2\lambda}} \right) b_{2\lambda} = \frac{1}{\pi} \int_{-\pi}^{\pi} g(x) sen\, \lambda x\, dx,$$

ou,

$$b_{1\lambda} = \frac{\frac{1}{\pi} \int_{-\pi}^{\pi} g(x) \cos \lambda x\, dx}{\left(\frac{1}{b^\lambda} - \frac{b^\lambda}{a^{2\lambda}} \right)},$$
$$b_{2\lambda} = \frac{\frac{1}{\pi} \int_{-\pi}^{\pi} g(x) sen\, \lambda x\, dx}{\left(\frac{1}{b^\lambda} - \frac{b^\lambda}{a^{2\lambda}} \right)}.$$

Com isso a solução u_2 fica,

$$\begin{aligned} u_2(\rho,\varphi) &= a_0 + b_0 \ln \rho \\ &+ \sum_{j=1} \left(\frac{1}{\rho^j} - \frac{\rho^j}{a^{2j}} \right) \cos j\varphi \frac{\frac{1}{\pi} \int_{-\pi}^{\pi} g(x) \cos jx\, dx}{\left(\frac{1}{b^j} - \frac{b^j}{a^{2j}} \right)} \\ &+ \sum_{j=1} \left(\frac{1}{\rho^j} - \frac{\rho^j}{a^{2j}} \right) sen\, j\varphi \frac{\frac{1}{\pi} \int_{-\pi}^{\pi} g(x) sen\, jx\, dx}{\left(\frac{1}{b^j} - \frac{b^j}{a^{2j}} \right)}, \end{aligned}$$

com,

$$a_0 = \frac{\ln a}{\ln a/b} \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x)\, dx\,,$$

$$b_0 = \frac{\dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} g(x)\, dx}{\ln b/a}\,.$$

Se $g(\varphi) = g_0$ *constante,*

$$u_2(\rho, \varphi) = a_0 + b_0 \ln \rho\,,$$

com,

$$a_0 = \frac{g_0 \ln a}{\ln a/b}\,,$$

$$b_0 = \frac{g_0}{\ln b/a}\,.$$

4. Considere o problema 1 com $u_a(\varphi) = \cos\varphi$.
O único coeficiente não nulo é $b_{11} = 1/a$*, logo a solução é,*

$$u(\rho, \varphi) = \frac{\rho}{a} \cos\, \varphi\,.$$

5. Considere o problema 1 com $u_a(\varphi) = \operatorname{sen}\varphi$.
Agora o único coeficiente não nulo é $b_{21} = 1/a$*, logo,*

$$u(\rho, \varphi) = \frac{\rho}{a} sen\, \varphi\,.$$

6. Considere o problema 3 com $u_a(\varphi) = \cos\varphi$ e $u_b(\varphi) = \operatorname{sen} 2\varphi$.
A solução é,

$$u(\rho, \varphi) = a_0 + b_0 \ln \rho + \sum_{j=1} \left(\frac{a_{1j}}{\rho^j} + a_{2j} \rho^j \right) [b_{1j} \cos j\varphi + b_{2j} sen\, j\varphi]\,,$$

que deve ficar na forma ($a_0 = b_0 = 0$)*,*

$$
\begin{aligned}
u(\rho,\varphi) \;&=\; \left(\frac{a_{11}}{\rho} + a_{21}\rho\right) b_{11}\cos\varphi \\
&\quad + \left(\frac{a_{12}}{\rho^2} + a_{22}\rho^2\right) b_{22} sen\, 2\varphi\,,
\end{aligned}
$$

com (os demais coeficientes se anulam),

$$
\begin{aligned}
\left(\frac{a_{11}}{a} + a_{21}a\right) b_{11} = 1\,, \\
\left(\frac{a_{12}}{a^2} + a_{22}a^2\right) b_{22} = 0\,, \\
\left(\frac{a_{11}}{b} + a_{21}b\right) b_{11} = 0\,, \\
\left(\frac{a_{12}}{b^2} + a_{22}b^2\right) b_{22} = 1\,.
\end{aligned}
$$

Portanto,

$$
\begin{aligned}
\frac{a_{12}}{a^2} + a_{22}a^2 = 0 \to a_{12} = -a_{22}a^4\,, \\
\frac{a_{11}}{b} + a_{21}b = 0 \to a_{11} = -a_{21}b^2\,,
\end{aligned}
$$

e,

$$
\begin{aligned}
a_{21}\left(\frac{-b^2 + a^2}{a}\right) b_{11} = 1\,, \\
a_{22}\left(\frac{-a^4 + b^4}{b^2}\right) b_{22} = 1\,.
\end{aligned}
$$

Podemos satisfazer o sistema acima escolhendo $a_{21} = a_{22} = 1$ *e temos então,*

$$
\begin{aligned}
b_{11} &= \frac{a}{a^2 - b^2}\,, \\
b_{22} &= \frac{b^2}{b^4 - a^4}\,, \\
a_{12} &= -a^4\,, \\
a_{11} &= -b^2\,.
\end{aligned}
$$

A solução é então,

$$\begin{aligned} u(\rho,\varphi) &= \left(-\frac{b^2}{\rho}+\rho\right)\frac{a}{a^2-b^2}\cos\varphi \\ &\quad+\left(-\frac{a^4}{\rho^2}+\rho^2\right)\frac{b^2}{b^4-a^4}sen\,2\varphi\,, \end{aligned}$$

7. Uma placa circular de raio a, com faces isoladas, possui metade do seu contorno a temperatura $f(\varphi)$ e a outra metade a temperatura $g(\varphi)$. Calcule a temperatura no estado estacionário (Spiegel [7], probl. 2.28 com $a = 1$ e $f = u_1$, $g = u_2$ constantes).

8. Uma placa circular de raio a, com faces isoladas, possui seu contorno com temperatura dada por $120+60\cos 2\varphi$. Calcule a temperatura no estado estacionário (Spiegel [7], probl. 2.53 com $a = 1$).

9. Uma placa circular de raio interno a e raio externo b possui faces isoladas, com temperaturas nos contornos $f(\varphi)$ e $g(\varphi)$, respectivamente. Calcule a temperatura no estado estacionário (Spiegel [7], probl. 2.67).

10. Considere o problema anterior nos limites (a) $a \to 0$, (b) $b \to \infty$ (Spiegel [7], probl. 2.68).

11. Uma placa possui a forma de um setor de círculo de raio a e ângulo β. Calcule a temperatura no estado estacionário se (Spiegel [7], probl. 2.76 com $g = h = 0$),

$$u(a,\varphi) = f(\varphi)\,,\quad u(\rho,0) = g(\rho)\,,\quad u(\rho,\beta) = h(\rho)\,.$$

12. Encontre a solução $u(\rho,\varphi)$ para as equações:

(a)

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2} = -f(\rho,\varphi)\,;$$

(b)

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{du}{d\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2} = +\alpha^2\,;$$

com α constante,

(c)

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{du}{d\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2} = -\alpha^2\,;$$

com α constante,

(d)

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{du}{d\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2} = +\alpha^2 u\,;$$

com α constante,

(e)

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{du}{d\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2}=-\alpha^2 u\,;$$

com α constante e,

$$0\le\rho\le a\,,\quad 0\le\varphi\le 2\pi\,,$$
$$u(a,\varphi)=\mu(\varphi)\,.$$

13. Encontre a solução $u(\rho,\varphi)$ para as equações:

(a)

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{du}{d\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2}=-f(\rho,\varphi)\,;$$

(b)

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{du}{d\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2}=+\alpha^2\,;$$

com α constante,

(c)

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{du}{d\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2}=-\alpha^2\,;$$

com α constante,

(d)

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{du}{d\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2}=+\alpha^2 u\,;$$

com α constante,

(e)

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{du}{d\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2}=-\alpha^2 u\,;$$

com α constante e,

$$a\le\rho\le b\,,\quad 0\le\varphi\le 2\pi\,,$$
$$u(a,\varphi)=\mu_1(\varphi)\,,\quad u(b,\varphi)=\mu_2(\varphi)\,.$$

7 Considerando $u = u(\rho, \varphi, z)$

Substituindo $u(\rho, \varphi, z) = R(\rho)\Phi(\varphi)Z(z)$ em (1), vem,

$$\frac{1}{\rho R}(\rho R')' + \frac{1}{\rho^2}\frac{\Phi''}{\Phi} + \frac{Z''}{Z} = 0\,.$$

Da equação acima temos,

$$\frac{1}{\rho R}(\rho R')' + \frac{1}{\rho^2}\frac{\Phi''}{\Phi} = -\frac{Z''}{Z} \equiv -\lambda^2\,. \qquad (45)$$

Podemos escrever a equação para Z,

$$Z'' - \lambda^2 Z = 0\,, \qquad (46)$$

Voltando a (45), temos,

$$\frac{1}{\rho R}(\rho R')' + \frac{1}{\rho^2}\frac{\Phi''}{\Phi} = -\lambda^2\,,$$

ou,

$$\frac{\rho}{R}(\rho R')' + \frac{\Phi''}{\Phi} + \lambda^2\rho^2 = 0\,.$$

Portanto,

$$\frac{\rho}{R}(\rho R')' + \lambda^2\rho^2 = -\frac{\Phi''}{\Phi} \equiv \mu^2\,.$$

Da relação acima temos,

$$\Phi'' + \mu^2\Phi = 0\,, \qquad (47)$$

$$\rho^2 R'' + \rho R' + (\lambda^2\rho^2 - \mu^2)R = 0\,. \qquad (48)$$

A equação (48) é a equação diferencial de Bessel, com solução,

$$R(\rho) = a_1 J_\mu(\lambda\rho) + a_2 Y_\mu(\lambda\rho)\,. \qquad (49)$$

As soluções acima são para $\lambda \neq 0$. Se $\lambda = 0$ temos para $R(\rho)$,

$$\rho^2 R'' + \rho R' - \mu^2 R = 0\,,$$

com solução,

$$R(\rho) = \frac{a_0}{\rho^\mu} + b_0\rho^\mu\,,$$

e para Z,

$$Z(z) = c_0 z + d_0\,.$$

Pelo princípio da superposição, a solução geral é uma soma das possíveis soluções individuais,

$$\begin{aligned}
u(\rho,\varphi,z) &= (c_0 z + d_0)(a_0 + b_0 \ln\rho) \\
&\quad + (e_0 z + f_0)\sum_{\mu}\left(\frac{a_{0\mu}}{\rho^{\mu}} + b_{0\mu}\rho^{\mu}\right)[d_{1\mu}\cos\mu\varphi + d_{2\mu}\mathrm{sen}\,\mu\varphi] \\
&\quad + \sum_{\lambda\mu}[a_{1\lambda\mu}J_{\mu}(\lambda\rho) + a_{2\lambda\mu}Y_{\mu}(\lambda\rho)]\times \\
&\quad \times[b_{1\mu}\cos\mu\varphi + b_{2\mu}\mathrm{sen}\,\mu\varphi][c_{1\lambda}\mathrm{cosh}\lambda z + c_{2\lambda}\mathrm{senh}\lambda z]\,, \qquad (50)
\end{aligned}$$

com u_0 constante. A determinação das constantes e dos possíveis valores de λ e μ depende das condições de contorno.

Consideramos agora outra possibilidade para a constante de separação λ. Escrevemos,

$$\frac{1}{\rho R}(\rho R')' + \frac{1}{\rho^2}\frac{\Phi''}{\Phi} = -\frac{Z''}{Z} \equiv +\lambda^2\,. \qquad (51)$$

Podemos escrever a equação para Z,

$$Z'' + \lambda^2 Z = 0\,, \qquad (52)$$

com solução,

$$Z(z) = c_{1\lambda}\cos\lambda z + c_{2\lambda}\mathrm{sen}\,\lambda z\,.$$

Voltando a (31), temos,

$$\frac{1}{\rho R}(\rho R')' + \frac{1}{\rho^2}\frac{\Phi''}{\Phi} = +\lambda^2\,,$$

ou,

$$\frac{\rho}{R}(\rho R')' + \frac{\Phi''}{\Phi} - \lambda^2\rho^2 = 0\,.$$

Portanto,

$$\frac{\rho}{R}(\rho R')' - \lambda^2\rho^2 = -\frac{\Phi''}{\Phi} \equiv \mu^2\,.$$

Da relação acima temos,

$$\Phi'' + \mu^2\Phi = 0\,, \tag{53}$$
$$\rho^2 R'' + \rho R' - (\lambda^2\rho^2 + \mu^2)R = 0\,. \tag{54}$$

A equação (54) é a equação diferencial modificada de Bessel, com solução,

$$R(\rho) = a_1 I_\mu(\lambda\rho) + a_2 K_\mu(\lambda\rho)\,, \quad \mu \neq 0\,. \tag{55}$$

Para $\mu = 0$ temos $R = a_0 + b_0 \ln\rho$. A solução geral é então,

$$\begin{aligned} u(\rho,\varphi,z) \;=\;& (c_0 z + d_0)(a_0 + b_0\ln\rho) && (56)\\ & +(e_0 z + f_0)\sum_\mu \left(\frac{a_{0\mu}}{\rho^\mu} + b_{0\mu}\rho^\mu\right)[d_{1\mu}\cos\mu\varphi + d_{2\mu}\text{sen}\,\mu\varphi] \\ & + \sum_{\mu\lambda}[a_{1\lambda\mu}I_\mu(\lambda\rho) + a_{2\lambda\mu}K_\mu(\lambda\rho)] \times \\ & \times[b_{1\mu}\cos\mu\varphi + b_{2\mu}\text{sen}\,\mu\varphi][c_{1\lambda}\cos\lambda z + c_{2\lambda}\text{sen}\,\lambda z]\,. && (57) \end{aligned}$$

A determinação das constantes e dos possíveis valores de λ e μ depende, como é usual, das condições de contorno.

8 Problemas

1. Considerando o intervalo $0 \leq \rho \leq a$, $0 \leq z \leq L$, $0 \leq \varphi \leq 2\pi$, encontre a solução $u(\rho,\varphi,z)$ com as condições de contorno,

$$u(a,\varphi,z) = f(\varphi,z)\,,$$
$$u(\rho,\varphi,0) = g(\rho,\varphi)\,,$$
$$u(\rho,\varphi,L) = h(\rho,\varphi)\,.$$

Como antes, escrevemos a solução geral como uma soma de soluções, cada uma satisfazendo uma das condições de contorno. Portanto,

$$u(\rho,\varphi,z) = u_1(\rho,\varphi,z) + u_2(\rho,\varphi,z) + u_3(\rho,\varphi,z)\,,$$

com,

$$u_1(a,\varphi,z) = f(\varphi,z)\,,\ \ u_1(\rho,\varphi,0) = 0\,,\ \ u_1(\rho,\varphi,L) = 0\,,$$
$$u_2(a,\varphi,z) = 0\,,\ \ u_2(\rho,\varphi,0) = g(\rho,\varphi)\,,\ \ u_2(\rho,\varphi,L) = 0\,,$$
$$u_3(a,\varphi,z) = 0\,,\ \ u_3(\rho,\varphi,0) = 0\,,\ \ u_3(\rho,\varphi,L) = h(\rho,\varphi)\,.$$

(a) Cálculo de u_1.
Escrevemos a solução como,

$$\begin{aligned} u_1(\rho,\varphi,z) &= (c_0 z + d_0)(a_0 + b_0 \ln \rho) \\ &+ (e_0 z + f_0) \sum_{\mu} \left(\frac{a_{0\mu}}{\rho^{\mu}} + b_{0\mu}\rho^{\mu} \right) [d_{1\mu} \cos \mu\varphi + d_{2\mu} sen\, \mu\varphi] \\ &+ \sum_{\mu\lambda} [a_{1\lambda\mu} I_{\mu}(\lambda\rho) + a_{2\lambda\mu} K_{\mu}(\lambda\rho)] \times \\ &\times [b_{1\mu} \cos \mu\varphi + b_{2\mu} sen\, \mu\varphi][c_{1\lambda} \cos \lambda z + c_{2\lambda} sen\, \lambda z]\,. \end{aligned}$$

A forma acima permite obter a expansão de $f(\varphi,z)$ em séries de Fourier. A solução finita em $\rho = 0$ é, fazendo $a_0 = b_{0\mu} = a_{1\lambda\mu} = 1$,

$$\begin{aligned} u_1(\rho,\varphi,z) &= c_0 z + d_0 \\ &+ (e_0 z + f_0) \sum_{\mu} \rho^{\mu} [d_{1\mu} \cos \mu\varphi + d_{2\mu} sen\, \mu\varphi] \\ &+ \sum_{\mu\lambda} I_{\mu}(\lambda\rho) [b_{1\mu} \cos \mu\varphi + b_{2\mu} sen\, \mu\varphi][c_{1\lambda} \cos \lambda z + c_{2\lambda} sen\, \lambda z]\,. \end{aligned}$$

As condições de contorno são,

$$\begin{aligned}
u_1(a,\varphi,z) &= f(\varphi,z) = c_0 z + d_0 \\
&+(e_0 z + f_0)\sum_{\mu} a^{\mu}[d_{1\mu}\cos\mu\varphi + d_{2\mu} sen\,\mu\varphi] \\
&+\sum_{\mu\lambda} I_{\mu}(\lambda a)[b_{1\mu}\cos\mu\varphi + b_{2\mu} sen\,\mu\varphi][c_{1\lambda}\cos\lambda z + c_{2\lambda} sen\,\lambda z]\,, \\
u_1(\rho,\varphi,0) &= 0 = d_0 + f_0\sum_{\mu}\rho^{\mu}[d_{1\mu}\cos\mu\varphi + d_{2\mu} sen\,\mu\varphi] \\
&+\sum_{\mu\lambda} I_{\mu}(\lambda\rho)[b_{1\mu}\cos\mu\varphi + b_{2\mu} sen\,\mu\varphi]c_{1\lambda}\,, \\
u_1(\rho,\varphi,L) &= 0 = c_0 L + d_0 + (e_0 L + f_0)\sum_{\mu}\rho^{\mu}[d_{1\mu}\cos\mu\varphi + d_{2\mu} sen\,\mu\varphi] \\
&+\sum_{\mu\lambda} I_{\mu}(\lambda\rho)[b_{1\mu}\cos\mu\varphi + b_{2\mu} sen\,\mu\varphi][c_{1\lambda}\cos\lambda L + c_{2\lambda} sen\,\lambda L]\,.
\end{aligned}$$

Satisfazemos as condições acima escolhendo,

$$\begin{aligned}
&c_0 = d_0 = e_0 = f_0 = 0 \\
&c_{1\lambda} = 0\,, \\
&\lambda_j L = \pi j\,,\quad j = 1, 2, \ldots
\end{aligned}$$

A condição para $f(\varphi, z)$ *é assim, fazendo* $c_{2\lambda} = 1$,

$$f(\varphi,z) = \sum_{\mu j} I_{\mu}(\lambda_j a)[b_{1\mu}\cos\mu\varphi + b_{2\mu} sen\,\mu\varphi] sen\,\lambda_j z\,.$$

Temos uma série de Fourier, logo,

$$\begin{aligned}
&\sum_{j} I_0(\lambda_j a) b_{10} sen\,\lambda_j z = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x,z)\,dx\,, \\
&\sum_{j} I_{\mu}(\lambda_j a) b_{1\mu} sen\,\lambda_j z = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x,z)\cos\mu x\,dx\,, \\
&\sum_{j} I_{\mu}(\lambda_j a) b_{2\mu} sen\,\lambda_j z = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x,z) sen\,\mu x\,dx\,,\quad \mu = 1, 2, \ldots
\end{aligned}$$

As expressões acima são expansões em série de Fourier de senos, logo,

$$
\begin{aligned}
I_0(\lambda_j a)b_{10} &= \frac{2}{L}\int_0^L sen\,(j\pi y/L)\,dy\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x,y)\,dx\,,\\
I_\mu(\lambda_j a)b_{1\mu} &= \frac{2}{L}\int_0^L sen\,(j\pi y/L)\,dy\frac{1}{\pi}\int_{-\pi}^{\pi} f(x,y)\cos\mu x\,dx\,,\\
I_\mu(\lambda_j a)b_{2\mu} &= \frac{2}{L}\int_0^L sen\,(j\pi y/L)\,dy\frac{1}{\pi}\int_{-\pi}^{\pi} f(x,y) sen\,\mu x\,dx\,,\quad \mu=1,2,\ldots
\end{aligned}
$$

Obtemos assim,

$$
\begin{aligned}
b_{10} &= \frac{1}{I_0(\lambda_j a)}\frac{2}{L}\int_0^L sen\,(j\pi y/L)\,dy\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x,y)\,dx\,,\\
b_{1\mu} &= \frac{1}{I_\mu(\lambda_j a)}\frac{2}{L}\int_0^L sen\,(j\pi y/L)\,dy\frac{1}{\pi}\int_{-\pi}^{\pi} f(x,y)\cos\mu x\,dx\,,\\
b_{2\mu} &= \frac{1}{I_\mu(\lambda_j a)}\frac{2}{L}\int_0^L sen\,(j\pi y/L)\,dy\frac{1}{\pi}\int_{-\pi}^{\pi} f(x,y) sen\,\mu x\,dx\,,\quad \mu=1,2,\ldots
\end{aligned}
$$

A série para $f(\varphi,z)$ é assim,

$$
\begin{aligned}
f(\varphi,z) &= \sum_j I_0(\lambda_j a)b_{10} sen\,\lambda_j z\\
&\quad+\sum_{\mu j} I_\mu(\lambda_j a)b_{1\mu}\cos\mu\varphi sen\,\lambda_j z\\
&\quad+\sum_{\mu j} I_\mu(\lambda_j a)b_{2\mu} sen\,\mu\varphi sen\,\lambda_j z\,,\\
&= \sum_j sen\,\lambda_j z\frac{2}{L}\int_0^L sen\,(j\pi y/L)\,dy\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x,y)\,dx\\
&\quad+\sum_{\mu j}\cos\mu\varphi sen\,\lambda_j z\frac{2}{L}\int_0^L sen\,(j\pi y/L)\,dy\frac{1}{\pi}\int_{-\pi}^{\pi} f(x,y)\cos\mu x\,dx\\
&\quad+\sum_{\mu j} sen\,\mu\varphi sen\,\lambda_j z\frac{2}{L}\int_0^L sen\,(j\pi y/L)\,dy\frac{1}{\pi}\int_{-\pi}^{\pi} f(x,y) sen\,\mu x\,dx\,,
\end{aligned}
$$

Se $f(\varphi,z)=f_0$ constante,

$$
\begin{aligned}
f_0 &= \sum_j sen\,\lambda_j z \frac{2f_0}{L}\int_0^L sen\,(j\pi z/L)\,dz\,,\\
&= \frac{4f_0}{\pi}\sum_j \frac{sen\,[(2j-1)\pi z/L]}{2j-1}\,,
\end{aligned}
$$

como esperado.

A solução u_1 *é então,*

$$
\begin{aligned}
u_1(\rho,\varphi,z) &= \sum_{j\mu} I_\mu(\lambda_j\rho)[b_{1\mu}\cos\mu\varphi + b_{2\mu} sen\,\mu\varphi] sen\,\lambda_j z\,,\\
&= \sum_j \frac{I_0(\lambda_j\rho)}{I_0(\lambda_j a)} sen\,\lambda_j z\,\times\\
&\quad \times\frac{2}{L}\int_0^L sen\,(j\pi y/L)\,dy \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x,y)\,dx\\
&\quad +\sum_{j\mu}\frac{I_\mu(\lambda_j\rho)}{I_\mu(\lambda_j a)}\cos\mu\varphi sen\,\lambda_j z\,\times\\
&\quad \times\frac{2}{L}\int_0^L sen\,(j\pi y/L)\,dy \frac{1}{\pi}\int_{-\pi}^{\pi} f(x,y)\cos\mu x\,dx\\
&\quad +\sum_{j\mu}\frac{I_\mu(\lambda_j\rho)}{I_\mu(\lambda_j a)} sen\,\mu\varphi sen\,\lambda_j z\,\times\\
&\quad \times\frac{2}{L}\int_0^L sen\,(j\pi y/L)\,dy \frac{1}{\pi}\int_{-\pi}^{\pi} f(x,y) sen\,\mu x\,dx\,.
\end{aligned}
$$

Se $f(\varphi,z) = f_0$ *constante,*

$$
\begin{aligned}
u_1(\rho,\varphi,z) &= \sum_j \frac{I_0(\lambda_j\rho)}{I_0(\lambda_j a)} sen\,\lambda_j z \frac{2f_0}{L}\int_0^L sen\,(j\pi y/L)\,dy\,,\\
&= \frac{2f_0}{L}\sum_j \frac{I_0(\lambda_j\rho)}{I_0(\lambda_j a)} sen\,\lambda_j z \int_0^L sen\,(j\pi y/L)\,dy\,,\\
&= \frac{2f_0}{L}\sum_j \frac{I_0((2j-1)\pi\rho/L)}{I_0((2j-1)\pi a/L)} sen\,(j\pi z/L)\frac{2L}{(2j-1)\pi}\,,\\
&= \frac{4f_0}{\pi}\sum_j \frac{I_0((2j-1)\pi\rho/L)}{I_0((2j-1)\pi a/L)}\frac{sen\,[(2j-1)\pi z/L]}{2j-1}\,.
\end{aligned}
$$

Se $f_0 = 0$ *temos* $u_1 = 0$.

(b) Cálculo de u_2.

Escrevemos,

$$\begin{aligned} u_2(\rho,\varphi,z) &= (c_0 z + d_0)(a_0 + b_0 \ln\rho) \\ &+(e_0 z + f_0)\sum_{\mu} \rho^{\mu}[d_{1\mu}\cos\mu\varphi + d_{2\mu} sen\,\mu\varphi] \\ &+\sum_{\lambda\mu} J_{\mu}(\lambda\rho)[b_{1\mu}\cos\mu\varphi + b_{2\mu} sen\,\mu\varphi] senh\lambda(L-z)\,, \end{aligned} \tag{58}$$

As condições de contorno nos dão,

$$\begin{aligned} u_2(a,\varphi,z) &= 0 = (c_0 z + d_0)(a_0 + b_0 \ln a) \\ &+(e_0 z + f_0)\sum_{\mu} a^{\mu}[d_{1\mu}\cos\mu\varphi + d_{2\mu} sen\,\mu\varphi] \\ &+\sum_{\lambda\mu} J_{\mu}(\lambda a)[b_{1\mu}\cos\mu\varphi + b_{2\mu} sen\,\mu\varphi] \times \\ &\times senh\lambda(L-z)]\,, \\ u_2(\rho,\varphi,0) &= g(\rho,\varphi) = d_0(a_0 + b_0\ln\rho) \\ &+f_0\sum_{\mu} \rho^{\mu}[d_{1\mu}\cos\mu\varphi + d_{2\mu} sen\,\mu\varphi] \\ &+\sum_{\lambda\mu} J_{\mu}(\lambda\rho)[b_{1\mu}\cos\mu\varphi + b_{2\mu} sen\,\mu\varphi] senh\lambda L\,, \\ u_2(\rho,\varphi,L) &= 0 = (c_0 L + d_0)(a_0 + b_0\ln\rho) \\ &+(e_0 L + f_0)\sum_{\mu} \rho^{\mu}[d_{1\mu}\cos\mu\varphi + d_{2\mu} sen\,\mu\varphi]\,. \end{aligned}$$

Satisfazemos as condições acima escolhendo,

$$\begin{aligned} &c_0 = d_0 = a_0 = b_0 = e_0 = f_0 = 0\,, \\ &J_{\mu}(\lambda_{\mu j} a) = 0\,, \quad \mu = 0,1,2,\ldots, \quad j = 1,2,\ldots \end{aligned}$$

logo,

$$u_2(\rho,\varphi,z) = \sum_{\mu j} J_{\mu}(\lambda_{\mu j}\rho)[b_{1\mu}\cos\mu\varphi + b_{2\mu} sen\,\mu\varphi] senh\,\lambda_{\mu j}(L-z)\,.$$

A condição para $g(\rho,\varphi)$ fica então,

$$g(\rho,\varphi) \;=\; \sum_{\mu j} J_\mu(\lambda_{\mu j}\rho)[b_{1\mu}\cos\mu\varphi + b_{2\mu} sen\,\mu\varphi] senh\,\lambda_{\mu j} L\,.$$

A expressão acima é uma série de Fourier, logo,

$$\sum_j J_0(\lambda_{0j}\rho) b_{10} senh\,\lambda_{0j} L = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\rho,y)\,dy\,,$$
$$\sum_j J_\mu(\lambda_{\mu j}\rho) b_{1\mu} senh\,\lambda_{\mu j} L = \frac{1}{\pi}\int_{-\pi}^{\pi} g(\rho,y)\cos\mu y\,dy\,,$$
$$\sum_j J_\mu(\lambda_{\mu j}\rho) b_{2\mu} senh\,\lambda_{j} L = \frac{1}{\pi}\int_{-\pi}^{\pi} g(\rho,y) sen\,\mu y\,dy\,,\quad \mu = 1,2,\ldots$$

As expressões acima são expansões em série de funções de Bessel, portanto,

$$b_{10} senh\,\lambda_{0j} L = \frac{2}{a^2 J_1^2(\lambda_{0j}a)}\int_0^a x J_0(\lambda_{0j}x)dx \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x,y)\,dy\,,$$
$$b_{1\mu} senh\,\lambda_{\mu j} L = \frac{2}{a^2 J_{\mu+1}^2(\lambda_{\mu j}a)}\int_0^a x J_\mu(\lambda_{\mu j}x)dx \frac{1}{\pi}\int_{-\pi}^{\pi} g(x,y)\cos\mu y\,dy\,,$$
$$b_{2\mu} senh\,\lambda_{\mu j} L = \frac{2}{a^2 J_{\mu+1}^2(\lambda_{\mu j}a)}\int_0^a x J_\mu(\lambda_{\mu j}x)dx \frac{1}{\pi}\int_{-\pi}^{\pi} g(x,y) sen\,\mu y\,dy\,,$$
$$\mu = 1,2,\ldots$$

ou,

$$b_{10} = \frac{1}{senh\,\lambda_{0j}L}\frac{2}{a^2 J_1^2(\lambda_{0j}a)}\int_0^a x J_0(\lambda_{0j}x)dx \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x,y)\,dy\,,$$
$$b_{1\mu} = \frac{1}{senh\,\lambda_{\mu j}L}\frac{2}{a^2 J_{\mu+1}^2(\lambda_{\mu j}a)}\int_0^a x J_\mu(\lambda_{\mu j}x)dx \frac{1}{\pi}\int_{-\pi}^{\pi} g(x,y)\cos\mu y\,dy\,,$$
$$b_{2\mu} = \frac{1}{senh\,\lambda_{\mu j}L}\frac{2}{a^2 J_{\mu+1}^2(\lambda_{\mu j}a)}\int_0^a x J_\mu(\lambda_{\mu j}x)dx \frac{1}{\pi}\int_{-\pi}^{\pi} g(x,y) sen\,\mu y\,dy\,,$$
$$\mu = 1,2,\ldots$$

A expansão para $g(\rho,\varphi)$ é assim,

$$
\begin{aligned}
g(\rho,\varphi) &= \sum_{\mu j} J_\mu(\lambda_{\mu j}\rho)[b_{1\mu}\cos\mu\varphi + b_{2\mu}sen\,\mu\varphi]senh\,\lambda_{\mu j}L \\
&= \sum_j J_0(\lambda_{0j}\rho)b_{10}senh\,\lambda_{0j}L \\
&\quad + \sum_{\mu j} J_\mu(\lambda_{\mu j}\rho)b_{1\mu}\cos\mu\varphi senh\,\lambda_{\mu j}L \\
&\quad + \sum_{\mu j} J_\mu(\lambda_{\mu j}\rho)b_{2\mu}sen\,\mu\varphi senh\,\lambda_{\mu j}L\,, \\
&= \frac{2}{a^2}\sum_j \frac{J_0(\lambda_{0j}\rho)}{J_1^2(\lambda_{0j}a)}\int_0^a xJ_0(\lambda_{0j}x)dx\frac{1}{2\pi}\int_{-\pi}^{\pi} g(x,y)\,dy \\
&\quad + \frac{2}{a^2}\sum_{\mu j}\frac{J_\mu(\lambda_{\mu j}\rho)}{J_{\mu+1}^2(\lambda_{\mu j}a)}\cos\mu\varphi \times \\
&\quad \times \int_0^a xJ_\mu(\lambda_{\mu j}x)dx\frac{1}{\pi}\int_{-\pi}^{\pi} g(x,y)\cos\mu y\,dy \\
&\quad + \frac{2}{a^2}\sum_{\mu j}\frac{J_\mu(\lambda_{\mu j}\rho)}{J_{\mu+1}^2(\lambda_{\mu j}a)}sen\,\mu\varphi \times \\
&\quad \times \int_0^a xJ_\mu(\lambda_{\mu j}x)dx\frac{1}{\pi}\int_{-\pi}^{\pi} g(x,y)sen\,\mu y\,dy\,.
\end{aligned}
$$

Se $g(\rho,\varphi) = g_0$ constante,

$$
\begin{aligned}
g_0 &= \frac{2}{a^2}\sum_j \frac{J_0(\lambda_{0j}\rho)}{J_1^2(\lambda_{0j}a)}\int_0^a xJ_0(\lambda_{0j}x)dx\,g_0\,, \\
&= \frac{2g_0}{a}\sum_j \frac{J_0(\lambda_{0j}\rho)}{\lambda_{0j}J_1(\lambda_{0j}a)} = g_0\,,
\end{aligned}
$$

como esperado.

A solução é então ([7], 6.97a),

$$
\begin{aligned}
u_2(\rho,\varphi,z) &= \frac{2}{a^2}\sum_j \frac{J_0(\lambda_{0j}\rho)}{J_1^2(\lambda_{0j}a)}\frac{senh\,\lambda_{0j}(L-z)}{senh\,\lambda_{0j}L}\times\\
&\times\int_0^a xJ_0(\lambda_{0j}x)dx\frac{1}{2\pi}\int_{-\pi}^{\pi} g(x,y)\,dy\\
&+\frac{2}{a^2}\sum_{\mu j}\frac{J_\mu(\lambda_{\mu j}\rho)}{J_{\mu+1}^2(\lambda_{\mu j}a)}\frac{senh\,\lambda_{\mu j}(L-z)}{senh\,\lambda_{\mu j}L}\cos\mu\varphi\times\\
&\times\int_0^a xJ_\mu(\lambda_{\mu j}x)dx\frac{1}{\pi}\int_{-\pi}^{\pi} g(x,y)\cos\mu y\,dy\\
&+\frac{2}{a^2}\sum_{\mu j}\frac{J_\mu(\lambda_{\mu j}\rho)}{J_{\mu+1}^2(\lambda_{\mu j}a)}\frac{senh\,\lambda_{\mu j}(L-z)}{senh\,\lambda_{\mu j}L}sen\,\mu\varphi\times\\
&\times\int_0^a xJ_\mu(\lambda_{\mu j}x)dx\frac{1}{\pi}\int_{-\pi}^{\pi} g(x,y)sen\,\mu y\,dy\,.
\end{aligned}
$$

Se $g(\rho,\varphi)=g_0$ *constante,*

$$
\begin{aligned}
u_2(\rho,\varphi,z) &= \frac{2}{a^2}\sum_j \frac{J_0(\lambda_{0j}\rho)}{J_1^2(\lambda_{0j}a)}\frac{senh\,\lambda_{0j}(L-z)}{senh\,\lambda_{0j}L}\times\\
&\times\int_0^a xJ_0(\lambda_{0j}x)dxg_0\frac{1}{2\pi}\int_{-\pi}^{\pi} dy\,,\\
&= \frac{2}{a^2}\sum_j \frac{J_0(\lambda_{0j}\rho)}{J_1^2(\lambda_{0j}a)}\frac{senh\,\lambda_{0j}(L-z)}{senh\,\lambda_{0j}L}\times\\
&\times g_0\int_0^a xJ_0(\lambda_{0j}x)dx\,,\\
&= \frac{2g_0}{a}\sum_j \frac{J_0(\lambda_{0j}\rho)}{\lambda_{0j}J_1(\lambda_{0j}a)}\frac{senh\,\lambda_{0j}(L-z)}{senh\,\lambda_{0j}L}\,.
\end{aligned}
$$

Se $g_0=0$ *temos* $u_2=0$.

Se $g(\rho,\varphi)=\rho^2\cos\varphi$ *([7], 6.97b),*

$$
\begin{aligned}
u_2(\rho,\varphi,z) &= \sum_j J_0(\lambda_j\rho) senh\,\lambda_j(L-z)\times\\
&\times\frac{1}{senh\,\lambda_j L}\frac{2}{a^2J_1^2(\lambda_j a)}\int_0^a xJ_0(\lambda_j x)dx\frac{1}{2\pi}\int_{-\pi}^{\pi}x^2\cos y\,dy\\
&+\sum_{\mu j} J_\mu(\lambda_j\rho)\cos\mu\varphi senh\,\lambda_j(L-z)\times\\
&\times\frac{1}{senh\,\lambda_j L}\frac{2}{a^2J_{\mu+1}^2(\lambda_j a)}\int_0^a xJ_\mu(\lambda_j x)dx\frac{1}{\pi}\int_{-\pi}^{\pi}x^2\cos y\cos\mu y\,dy\\
&+\sum_{\mu j} J_\mu(\lambda_j\rho) sen\,\mu\varphi senh\,\lambda_j(L-z)\times\\
&\times\frac{1}{senh\,\lambda_j L}\frac{2}{a^2J_{\mu+1}^2(\lambda_j a)}\int_0^a xJ_\mu(\lambda_j x)dx\frac{1}{\pi}\int_{-\pi}^{\pi}x^2\cos y sen\,\mu y\,dy\,.
\end{aligned}
$$

A última integral em φ se anula pois o integrando é ímpar. A segunda integral é nula para $\mu \neq 1$, e para $\mu = 1$ é igual a π. A primeira integral é nula. Portanto,

$$
\begin{aligned}
u_2(\rho,\varphi,z) &= \sum_j J_1(\lambda_j\rho)\cos\varphi senh\,\lambda_j(L-z)\times\\
&\times\frac{1}{senh\,\lambda_j L}\frac{2}{a^2J_2^2(\lambda_j a)}\int_0^a x^3J_1(\lambda_j x)dx\,,
\end{aligned}
$$

com $J_1(\lambda_j a) = 0$. Calculando a integral acima temos,

$$
\int_0^a x^3 J_1(\lambda_j x)dx = \frac{1}{\lambda_j^3}\left[(3a-\lambda_j^2a^3)J_0(\lambda_j a) - 3\int_0^a J_0(\lambda_j x)dx\right].
$$

Usamos as seguintes integrais indefinidas,

$$
\begin{aligned}
&\int \cos px\cos qx dx = \frac{sen(p-q)x}{2(p-q)} + \frac{sen(p+q)x}{2(p+q)}\,,\\
&\int senpx\cos qx dx = -\frac{\cos(p-q)x}{2(p-q)} - \frac{\cos(p+q)x}{2(p+q)}\,,\\
&\int \cos^2 ax dx = \frac{x}{2} + \frac{sen\,2ax}{4a}\,,\\
&\int x^m J_1(x)dx = -x^mJ_0(x) + m\int x^{m-1}J_0(x)dx\,,\\
&\int x^2J_0(x)dx = x^2J_1(x) + xJ_0(x) - \int J_0(x)dx\,.
\end{aligned}
$$

(c) Cálculo de u_3. Escrevemos,

$$u_3(\rho, \varphi, z) \;=\; \sum_{\lambda\mu} J_\mu(\lambda\rho)[b_{1\mu}\cos\mu\varphi + b_{2\mu} sen\,\mu\varphi][c_{1\lambda} cosh\lambda z + c_{2\lambda} senh\lambda z]\,.$$

As condições de contorno são,

$$\begin{aligned}
u_3(a, \varphi, z) &= 0 = \sum_{\lambda\mu} J_\mu(\lambda a)[b_{1\mu}\cos\mu\varphi + b_{2\mu} sen\,\mu\varphi] \times \\
&\quad \times[c_{1\lambda} cosh\lambda z + c_{2\lambda} senh\lambda z]\,, \\
u_3(\rho, \varphi, 0) &= 0 = \sum_{\lambda\mu} J_\mu(\lambda\rho)[b_{1\mu}\cos\mu\varphi + b_{2\mu} sen\,\mu\varphi] c_{1\lambda}\,, \\
u_3(\rho, \varphi, L) &= h(\rho, \varphi) = \sum_{\lambda\mu} J_\mu(\lambda\rho)[b_{1\mu}\cos\mu\varphi + b_{2\mu} sen\,\mu\varphi] \times \\
&\quad \times[c_{1\lambda} cosh\lambda L + c_{2\lambda} senh\lambda L]\,.
\end{aligned}$$

Satisfazemos as condições acima escolhendo,

$$\begin{aligned}
&J_\mu(\lambda_{\mu j} a) = 0\,, \quad j = 1, 2, \ldots \\
&c_{1\lambda} = 0\,.
\end{aligned}$$

A solução fica então, fazendo $c_{2\lambda} = 1$,

$$u_3(\rho, \varphi, z) \;=\; \sum_{j\mu} J_\mu(\lambda_{\mu j}\rho)[b_{1\mu}\cos\mu\varphi + b_{2\mu} sen\,\mu\varphi] senh\lambda_{\mu j} z\,.$$

A condição para $h(\rho, \varphi)$ fica,

$$h(\rho, \varphi) \;=\; \sum_{j\mu} J_\mu(\lambda_{\mu j}\rho)[b_{1\mu}\cos\mu\varphi + b_{2\mu} sen\,\mu\varphi] senh\lambda_{\mu j} L\,,$$

Temos uma série de Fourier para $h(\rho, \varphi)$, logo,

$$\begin{aligned}
\sum_j J_0(\lambda_{0j}\rho) b_{10} senh\lambda_{0j} L &= \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\rho, y)\, dy\,, \\
\sum_j J_\mu(\lambda_{\mu j}\rho) b_{1\mu} senh\lambda_{\mu j} L &= \frac{1}{\pi}\int_{-\pi}^{\pi} h(\rho, y)\cos\mu y\, dy\,, \\
\sum_j J_\mu(\lambda_{\mu j}\rho) b_{2\mu} senh\lambda_{\mu j} L &= \frac{1}{\pi}\int_{-\pi}^{\pi} h(\rho, y) sen\,\mu y\, dy\,.
\end{aligned}$$

As expressões acima são expansões em série de funções de Bessel, portanto,

$$
\begin{aligned}
b_{10} &= \frac{2}{senh\lambda_{0j}L\, a^2 J_1^2(\lambda_{0j}a)} \int_0^a xJ_0(\lambda_{0j}x)dx \frac{1}{2\pi}\int_{-\pi}^{\pi} h(x,y)\, dy\,, \\
b_{1\mu} &= \frac{2}{senh\lambda_{\mu j}L\, a^2 J_{\mu+1}^2(\lambda_{\mu j}a)} \int_0^a xJ_\mu(\lambda_{\mu j}x)dx \frac{1}{\pi}\int_{-\pi}^{\pi} h(x,y)\cos\mu y\, dy\,, \\
b_{2\mu} &= \frac{2}{senh\lambda_{\mu j}L\, a^2 J_{\mu+1}^2(\lambda_{\mu j}a)} \int_0^a xJ_\mu(\lambda_{\mu j}x)dx \frac{1}{\pi}\int_{-\pi}^{\pi} h(x,y) sen\,\mu y\, dy\,.
\end{aligned}
$$

A série para $h(\rho,\varphi)$ *é assim,*

$$
\begin{aligned}
h(\rho,\varphi) \;=\; & \frac{2}{a^2}\sum_j \frac{J_0(\lambda_{0j}\rho)}{J_1^2(\lambda_{0j}a)} \int_0^a xJ_0(\lambda_{0j}x)dx \frac{1}{2\pi}\int_{-\pi}^{\pi} h(x,y)\, dy \\
& + \frac{2}{a^2}\sum_{j\mu} \frac{J_\mu(\lambda_{\mu j}\rho)}{J_{\mu+1}^2(\lambda_{\mu j}a)} \cos\mu\varphi \times \\
& \times \int_0^a xJ_\mu(\lambda_{\mu j}x)dx \frac{1}{\pi}\int_{-\pi}^{\pi} h(x,y)\cos\mu y\, dy \\
& + \frac{2}{a^2}\sum_{j\mu} \frac{J_\mu(\lambda_{\mu j}\rho)}{J_{\mu+1}^2(\lambda_{\mu j}a)} sen\,\mu\varphi \times \\
& \times \int_0^a xJ_\mu(\lambda_{\mu j}x)dx \frac{1}{\pi}\int_{-\pi}^{\pi} h(x,y) sen\,\mu y\, dy\,.
\end{aligned}
$$

Se $h(\rho,\varphi) = h_0$ *constante,*

$$
h(\rho,\varphi) \;=\; \frac{2h_0}{a}\sum_j \frac{J_0(\lambda_{0j}\rho)}{\lambda_{0j}J_1(\lambda_{0j}a)} = h_0\,,
$$

como esperado.

A solução é então,

$$
\begin{aligned}
u_3(\rho,\varphi,z) &= \frac{2}{a^2}\sum_j \frac{J_0(\lambda_{0j}\rho)}{J_1^2(\lambda_{0j}a)}\frac{senh\lambda_{0j}z}{senh\lambda_{0j}L}\times \\
&\times\int_0^a xJ_0(\lambda_{0j}x)dx\frac{1}{2\pi}\int_{-\pi}^{\pi}h(x,y)\,dy \\
&+\frac{2}{a^2}\sum_{j\mu}\frac{J_\mu(\lambda_{\mu j}\rho)}{J_{\mu+1}^2(\lambda_{\mu j}a)}\frac{senh\lambda_{\mu j}z}{senh\lambda_{\mu j}L}\cos\mu\varphi\times \\
&\times\int_0^a xJ_\mu(\lambda_{\mu j}x)dx\frac{1}{\pi}\int_{-\pi}^{\pi}h(x,y)\cos\mu y\,dy \\
&+\frac{2}{a^2}\sum_{j\mu}\frac{J_\mu(\lambda_{\mu j}\rho)}{J_{\mu+1}^2(\lambda_{\mu j}a)}\frac{senh\lambda_{\mu j}z}{senh\lambda_{\mu j}L}sen\,\mu\varphi\times \\
&\times\int_0^a xJ_\mu(\lambda_{\mu j}x)dx\frac{1}{\pi}\int_{-\pi}^{\pi}h(x,y)sen\,\mu y\,dy\,.
\end{aligned}
$$

Se $h(\rho,\varphi)=h_0$ *constante,*

$$
\begin{aligned}
u_3(\rho,\varphi,z) &= \frac{2}{a^2}\sum_j \frac{J_0(\lambda_{0j}\rho)}{J_1^2(\lambda_{0j}a)}\frac{senh\lambda_{0j}z}{senh\lambda_{0j}L}h_0\int_0^a xJ_0(\lambda_{0j}x)dx\,, \\
&= \frac{2h_0}{a}\sum_j \frac{J_0(\lambda_{0j}\rho)}{\lambda_{0j}J_1(\lambda_{0j}a)}\frac{senh\lambda_{0j}z}{senh\lambda_{0j}L}\,.
\end{aligned}
$$

Se $h_0=0$ *temos* $u_3=0$.

(d) Considerando as funções nas condições de contorno constantes, a solução é,

$$
\begin{aligned}
u(\rho,\varphi,z) &= \frac{4f_0}{\pi}\sum_j \frac{I_0((2j-1)\pi\rho/L)}{I_0((2j-1)\pi a/L)}\frac{sen\,((2j-1)\pi z/L)}{2j-1} \\
&+\frac{2g_0}{a}\sum_j \frac{J_0(\lambda_{0j}\rho)}{\lambda_{0j}J_1(\lambda_{0j}a)}\frac{senh\,\lambda_{0j}(L-z)}{senh\,\lambda_{0j}L} \\
&+\frac{2h_0}{a}\sum_j \frac{J_0(\lambda_{0j}\rho)}{\lambda_{0j}J_1(\lambda_{0j}a)}\frac{senh\lambda_{0j}z}{senh\lambda_{0j}L}\,.
\end{aligned}
$$

2. Considerando o intervalo $a\le\rho\le b$, $0\le z\le L$, $0\le\varphi\le 2\pi$, encontre a solução $u(\rho,\varphi,z)$ com as condições de contorno,

$$
\begin{aligned}
u(a,\varphi,z) &= f(\varphi,z)\,,\\
u(b,\varphi,z) &= g(\varphi,z)\,,\\
u(\rho,\varphi,0) &= h(\rho,\varphi)\,,\\
u(\rho,\varphi,L) &= v(\rho,\varphi)\,.
\end{aligned}
$$

Escrevemos a solução como uma soma de soluções, cada uma satisfazendo uma das condições de contorno,

$$u(\rho,\varphi,z) = u_1(\rho,\varphi,z) + u_2(\rho,\varphi,z) + u_3(\rho,\varphi,z) + u_4(\rho,\varphi,z)\,,$$

com,

$$
\begin{aligned}
&u_1(a,\varphi,z) = f(\varphi,z)\,,\ \ u_1(b,\varphi,z) = 0\,,\ \ u_1(\rho,\varphi,0) = 0\,,\ \ u_1(\rho,\varphi,L) = 0\,,\\
&u_2(a,\varphi,z) = 0\,,\ \ u_2(b,\varphi,z) = g(\varphi,z)\,,\ \ u_2(\rho,\varphi,0) = 0\,,\ \ u_2(\rho,\varphi,L) = 0\,,\\
&u_3(a,\varphi,z) = 0\,,\ \ u_3(b,\varphi,z) = 0\,,\ \ u_3(\rho,\varphi,0) = h(\rho,\varphi)\,,\ \ u_3(\rho,\varphi,L) = 0\,,\\
&u_4(a,\varphi,z) = 0\,,\ \ u_4(b,\varphi,z) = 0\,,\ \ u_4(\rho,\varphi,0) = 0\,,\ \ u_4(\rho,\varphi,L) = v(\rho,\varphi)\,.
\end{aligned}
$$

(a) Cálculo de u_1.

Escrevemos a solução como,

$$
\begin{aligned}
u_1(\rho,\varphi,z) &= \sum_{\mu,\lambda>0}[a_{1\lambda}I_\mu(\lambda\rho) + a_{2\lambda}K_\mu(\lambda\rho)]\times\\
&\quad \times[b_{1\mu}\cos\mu\varphi + b_{2\mu}sen\,\mu\varphi][c_{1\lambda}\cos\lambda z + c_{2\lambda}sen\,\lambda z]\,.
\end{aligned}
$$

As condições de contorno são,

$$
\begin{aligned}
u_1(a,\varphi,z) &= f(\varphi,z) = \sum_{\mu,\lambda>0}[a_{1\lambda}I_\mu(\lambda a) + a_{2\lambda}K_\mu(\lambda a)]\times\\
&\quad \times[b_{1\mu}\cos\mu\varphi + b_{2\mu}sen\,\mu\varphi][c_{1\lambda}\cos\lambda z + c_{2\lambda}sen\,\lambda z]\,,\\
u_1(b,\varphi,z) &= 0 = \sum_{\mu,\lambda>0}[a_{1\lambda}I_\mu(\lambda b) + a_{2\lambda}K_\mu(\lambda b)]\times\\
&\quad \times[b_{1\mu}\cos\mu\varphi + b_{2\mu}sen\,\mu\varphi][c_{1\lambda}\cos\lambda z + c_{2\lambda}sen\,\lambda z]\,,\\
u_1(\rho,\varphi,0) &= 0 = \sum_{\mu,\lambda>0}[a_{1\lambda}I_\mu(\lambda\rho) + a_{2\lambda}K_\mu(\lambda\rho)]\times\\
&\quad \times[b_{1\mu}\cos\mu\varphi + b_{2\mu}sen\,\mu\varphi]c_{1\lambda}\,,\\
u_1(\rho,\varphi,L) &= 0 = \sum_{\mu,\lambda>0}[a_{1\lambda}I_\mu(\lambda\rho) + a_{2\lambda}K_\mu(\lambda\rho)]\times\\
&\quad \times[b_{1\mu}\cos\mu\varphi + b_{2\mu}sen\,\mu\varphi][c_{1\lambda}\cos\lambda L + c_{2\lambda}sen\,\lambda L]\,.
\end{aligned}
$$

Satisfazemos as condições acima escolhendo,

$$
\begin{aligned}
&a_{1\lambda} I_\mu(\lambda b) + a_{2\lambda} K_\mu(\lambda b) = 0\,,\\
&c_{1\lambda} = 0\,,\\
&\lambda_j L = \pi j\,, \quad j = 1, 2, \ldots
\end{aligned}
$$

Portanto,

$$a_{2j} = -a_{1j}\frac{I_\mu(\lambda_j b)}{K_\mu(\lambda_j b)}\,,$$

e a solução fica, fazendo $a_{1j} = c_{2\lambda} = 1$*,*

$$
\begin{aligned}
u_1(\rho,\varphi,z) &= \sum_{\mu j}[I_\mu(\lambda_j\rho)K_\mu(\lambda_j b) - I_\mu(\lambda_j b)K_\mu(\lambda_j \rho)]\frac{1}{K_\mu(\lambda_j b)} \times\\
&\quad \times[b_{1\mu}\cos\mu\varphi + b_{2\mu} sen\,\mu\varphi] sen\,\lambda_j z\,,
\end{aligned}
$$

ou,

$$u_1(\rho,\varphi,z) = \sum_{\mu j}\frac{V_\mu(\lambda_j\rho)}{K_\mu(\lambda_j b)}[b_{1\mu}\cos\mu\varphi + b_{2\mu} sen\,\mu\varphi] sen\,\lambda_j z\,.$$

A condição de contorno para $f(\varphi, z)$ *é então,*

$$f(\varphi,z) = \sum_{\mu j}\frac{V_\mu(\lambda_j a)}{K_\mu(\lambda_j b)}[b_{1\mu}\cos\mu\varphi + b_{2\mu} sen\,\mu\varphi] sen\,\lambda_j z\,.$$

Temos então uma série de Fourier para $f(\varphi, z)$*, logo,*

$$
\begin{aligned}
&\sum_j \frac{V_0(\lambda_j a)}{K_0(\lambda_j b)} b_{10} sen\,\lambda_j z = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x,z)dx\,,\\
&\sum_j \frac{V_\mu(\lambda_j a)}{K_\mu(\lambda_j b)} b_{1\mu} sen\,\lambda_j z = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x,z)\cos\mu x\,dx\,,\\
&\sum_{\mu j} \frac{V_\mu(\lambda_j a)}{K_\mu(\lambda_j b)} b_{2\mu} sen\,\lambda_j z = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x,z) sen\,\mu x\,dx\,, \quad \mu = 1, 2, \ldots
\end{aligned}
$$

As expressões acima são expansões em série de Fourier de senos, logo,

$$
\begin{aligned}
&\frac{V_0(\lambda_j a)}{K_0(\lambda_j b)} b_{10} = \frac{2}{L}\int_0^L sen\,(j\pi y/L)\,dy \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x,y)dx\,,\\
&\frac{V_\mu(\lambda_j a)}{K_\mu(\lambda_j b)} b_{1\mu} = \frac{2}{L}\int_0^L sen\,(j\pi y/L)\,dy \frac{1}{\pi}\int_{-\pi}^{\pi} f(x,y)\cos\mu x\,dx\,,\\
&\frac{V_\mu(\lambda_j a)}{K_\mu(\lambda_j b)} b_{2\mu} = \frac{2}{L}\int_0^L sen\,(j\pi y/L)\,dy \frac{1}{\pi}\int_{-\pi}^{\pi} f(x,y) sen\,\mu x\,dx\,,\\
&\mu = 1,2,\ldots
\end{aligned}
$$

ou,

$$
\begin{aligned}
&b_{10} = \frac{K_0(\lambda_j b)}{V_0(\lambda_j a)} \frac{2}{L}\int_0^L sen\,(j\pi y/L)\,dy \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x,y)dx\,,\\
&b_{1\mu} = \frac{K_\mu(\lambda_j b)}{V_\mu(\lambda_j a)} \frac{2}{L}\int_0^L sen\,(j\pi y/L)\,dy \frac{1}{\pi}\int_{-\pi}^{\pi} f(x,y)\cos\mu x\,dx\,,\\
&b_{2\mu} = \frac{K_\mu(\lambda_j b)}{V_\mu(\lambda_j a)} \frac{2}{L}\int_0^L sen\,(j\pi y/L)\,dy \frac{1}{\pi}\int_{-\pi}^{\pi} f(x,y) sen\,\mu x\,dx\,,\\
&\mu = 1,2,\ldots
\end{aligned}
$$

A série para $f(\varphi, z)$ *é assim,*

$$
\begin{aligned}
f(\varphi,z) \;=\; & \sum_j sen\,(j\pi z/L)\,\times \\
& \times \frac{2}{L}\int_0^L sen\,(j\pi y/L)\,dy \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x,y)dx \\
& + \sum_{\mu j} \cos\mu\varphi\, sen\,(j\pi z/L)\,\times \\
& \times \frac{2}{L}\int_0^L sen\,(j\pi y/L)\,dy \frac{1}{\pi}\int_{-\pi}^{\pi} f(x,y)\cos\mu x\,dx \\
& + \sum_{\mu j} sen\,\mu\varphi\, sen\,(j\pi z/L)\,\times \\
& \times \frac{2}{L}\int_0^L sen\,(j\pi y/L)\,dy \frac{1}{\pi}\int_{-\pi}^{\pi} f(x,y) sen\,\mu x\,dx\,.
\end{aligned}
$$

Se $f(\varphi, z) = f_0$ *constante,*

$$
\begin{aligned}
f_0 &= \sum_j sen\,(j\pi z/L)\frac{2}{L}f_0\int_0^L sen\,(j\pi y/L)\,dy\,, \\
&= \frac{4f_0}{\pi}\sum_j \frac{sen\,[(2j-1)\pi z/L]}{2j-1} = f_0\,,
\end{aligned}
$$

como esperado.

A solução u_1 é então,

$$
\begin{aligned}
u_1(\rho,\varphi,z) &= \sum_j \frac{V_0(j\pi\rho/L)}{V_0(j\pi a/L)} sen\,(j\pi z/L)\,\times \\
&\times \frac{2}{L}\int_0^L sen\,(j\pi y/L)\,dy\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x,y)dx \\
&+ \sum_{\mu j}\frac{V_\mu(j\pi\rho/L)}{V_\mu(j\pi a/L)}\cos\mu\varphi sen\,(j\pi z/L)\,\times \\
&\times \frac{2}{L}\int_0^L sen\,(j\pi y/L)\,dy\frac{1}{\pi}\int_{-\pi}^{\pi} f(x,y)\cos\mu x\,dx \\
&+ \sum_{\mu j}\frac{V_\mu(j\pi\rho/L)}{V_\mu(j\pi a/L)} sen\,\mu\varphi sen\,(j\pi z/L)\,\times \\
&\times \frac{2}{L}\int_0^L sen\,(j\pi y/L)\,dy\frac{1}{\pi}\int_{-\pi}^{\pi} f(x,y) sen\,\mu x\,dx\,.
\end{aligned}
$$

Se $f(\varphi,z)=f_0$ constante,

$$
\begin{aligned}
u_1(\rho,\varphi,z) &= \sum_j \frac{V_0(j\pi\rho/L)}{V_0(j\pi a/L)} sen\,(j\pi z/L)\frac{2}{L}f_0\int_0^L sen\,(j\pi y/L)\,dy\,, \\
&= \frac{4f_0}{\pi}\sum_j \frac{V_0((2j-1)\pi\rho/L)}{V_0((2j-1)\pi a/L)}\frac{sen\,[(2j-1)\pi z/L]}{2j-1}\,.
\end{aligned}
$$

Se $f_0 = 0$ temos $u_1 = 0$.

(b) Cálculo de u_2.

Escrevemos u_2 como,

$$
\begin{aligned}
u_2(\rho,\varphi,z) &= \sum_{\mu,\lambda>0}[a_{1\lambda}I_\mu(\lambda\rho) + a_{2\lambda}K_\mu(\lambda\rho)]\,\times \\
&\times [b_{1\mu}\cos\mu\varphi + b_{2\mu} sen\,\mu\varphi][c_{1\lambda}\cos\lambda z + c_{2\lambda} sen\,\lambda z]\,.
\end{aligned}
$$

As condições de contorno são

$$\begin{aligned}
u_2(a,\varphi,z) &= 0 = \sum_{\mu,\lambda>0}[a_{1\lambda}I_\mu(\lambda a)+a_{2\lambda}K_\mu(\lambda a)]\times \\
&\times[b_{1\mu}\cos\mu\varphi+b_{2\mu}sen\,\mu\varphi][c_{1\lambda}\cos\lambda z+c_{2\lambda}sen\,\lambda z]\,, \\
u_2(b,\varphi,z) &= g(\varphi,z) = \sum_{\mu,\lambda>0}[a_{1\lambda}I_\mu(\lambda b)+a_{2\lambda}K_\mu(\lambda b)]\times \\
&\times[b_{1\mu}\cos\mu\varphi+b_{2\mu}sen\,\mu\varphi][c_{1\lambda}\cos\lambda z+c_{2\lambda}sen\,\lambda z]\,, \\
u_2(\rho,\varphi,0) &= 0 = \sum_{\mu,\lambda>0}[a_{1\lambda}I_\mu(\lambda\rho)+a_{2\lambda}K_\mu(\lambda\rho)]\times \\
&\times[b_{1\mu}\cos\mu\varphi+b_{2\mu}sen\,\mu\varphi]c_{1\lambda}\,, \\
u_2(\rho,\varphi,L) &= 0 = \sum_{\mu,\lambda>0}[a_{1\lambda}I_\mu(\lambda\rho)+a_{2\lambda}K_\mu(\lambda\rho)]\times \\
&\times[b_{1\mu}\cos\mu\varphi+b_{2\mu}sen\,\mu\varphi][c_{1\lambda}\cos\lambda L+c_{2\lambda}sen\,\lambda L]\,,
\end{aligned}$$

Satisfazemos as condições acima escolhendo,

$$\begin{aligned}
&a_{1\lambda}I_\mu(\lambda a)+a_{2\lambda}K_\mu(\lambda a)=0 \Rightarrow a_{2\lambda}=-a_{1\lambda}\frac{I_\mu(\lambda a)}{K_\mu(\lambda a)}\,, \\
&c_{1\lambda}=0\,, \\
&\lambda_j L=\pi j\,,\ \ j=1,2,\ldots
\end{aligned}$$

A solução fica então, fazendo $a_{1j}=c_{2j}=1$,

$$\begin{aligned}
u_2(\rho,\varphi,z) &= \sum_{\mu j}[I_\mu(\lambda_j\rho)K_\mu(\lambda_j a)-I_\mu(\lambda_j a)K_\mu(\lambda_j\rho)]\frac{1}{K_\mu(\lambda_j a)}\times \\
&\times[b_{1\mu}\cos\mu\varphi+b_{2\mu}sen\,\mu\varphi]sen\,\lambda_j z\,,
\end{aligned}$$

ou,

$$u_2(\rho,\varphi,z) = \sum_{\mu j}\frac{W_\mu(\lambda_j\rho)}{K_\mu(\lambda_j a)}[b_{1\mu}\cos\mu\varphi+b_{2\mu}sen\,\mu\varphi]sen\,\lambda_j z\,.$$

A condição de contorno para g *então nos dá*

$$g(\varphi,z) = \sum_{\mu j}\frac{W_\mu(\lambda_j b)}{K_\mu(\lambda_j a)}[b_{1\mu}\cos\mu\varphi+b_{2\mu}sen\,\mu\varphi]sen\,\lambda_j z\,.$$

Temos uma série de Fourier para $g(\varphi, z)$*, logo,*

$$\sum_j \frac{W_0(\lambda_j b)}{K_0(\lambda_j a)} b_{10} sen\, \lambda_j z = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x,z) dx\,,$$
$$\sum_j \frac{W_\mu(\lambda_j b)}{K_\mu(\lambda_j a)} b_{1\mu} sen\, \lambda_j z = \frac{1}{\pi} \int_{-\pi}^{\pi} g(x,z) \cos \mu x\, dx\,,$$
$$\sum_j \frac{W_\mu(\lambda_j b)}{K_\mu(\lambda_j a)} b_{2\mu} sen\, \lambda_j z = \frac{1}{\pi} \int_{-\pi}^{\pi} g(x,z) sen\, \mu x\, dx\,, \quad \mu = 1, 2, \ldots$$

As expressões acima são séries de Fourier de senos, assim,

$$\frac{W_0(\lambda_j b)}{K_0(\lambda_j a)} b_{10} = \frac{2}{L} \int_0^L sen\,(j\pi y/L)\, dy \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x,y) dx\,,$$
$$\frac{W_\mu(\lambda_j b)}{K_\mu(\lambda_j a)} b_{1\mu} = \frac{2}{L} \int_0^L sen\,(j\pi y/L)\, dy \frac{1}{\pi} \int_{-\pi}^{\pi} g(x,y) \cos \mu x\, dx\,,$$
$$\frac{W_\mu(\lambda_j b)}{K_\mu(\lambda_j a)} b_{2\mu} = \frac{2}{L} \int_0^L sen\,(j\pi y/L)\, dy \frac{1}{\pi} \int_{-\pi}^{\pi} g(x,y) sen\, \mu x\, dx\,, \quad \mu = 1, 2, \ldots$$

ou,

$$b_{10} = \frac{K_0(\lambda_j a)}{W_0(\lambda_j b)} \frac{2}{L} \int_0^L sen\,(j\pi y/L)\, dy \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x,y) dx\,,$$
$$b_{1\mu} = \frac{K_\mu(\lambda_j a)}{W_\mu(\lambda_j b)} \frac{2}{L} \int_0^L sen\,(j\pi y/L)\, dy \frac{1}{\pi} \int_{-\pi}^{\pi} g(x,y) \cos \mu x\, dx\,,$$
$$b_{2\mu} = \frac{K_\mu(\lambda_j a)}{W_\mu(\lambda_j b)} \frac{2}{L} \int_0^L sen\,(j\pi y/L)\, dy \frac{1}{\pi} \int_{-\pi}^{\pi} g(x,y) sen\, \mu x\, dx\,, \quad \mu = 1, 2, \ldots$$

Com isso a solução u_2 *é,*

$$
\begin{aligned}
u_2(\rho,\varphi,z) &= \sum_j \frac{W_0(\lambda_j\rho)}{W_0(\lambda_j b)} sen\,\lambda_j z \times \\
&\times \frac{2}{L}\int_0^L sen\,(j\pi y/L)\,dy \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x,y)dx \\
&+ \sum_{\mu j} \frac{W_\mu(\lambda_j\rho)}{W_\mu(\lambda_j b)} \cos\mu\varphi sen\,\lambda_j z \times \\
&\times \frac{2}{L}\int_0^L sen\,(j\pi y/L)\,dy \frac{1}{\pi}\int_{-\pi}^{\pi} g(x,y)\cos\mu x\,dx \\
&+ \sum_{\mu j} \frac{W_\mu(\lambda_j\rho)}{W_\mu(\lambda_j b)} sen\,\mu\varphi sen\,\lambda_j z \times \\
&\times \frac{2}{L}\int_0^L sen\,(j\pi y/L)\,dy \frac{1}{\pi}\int_{-\pi}^{\pi} g(x,y) sen\,\mu x\,dx\,.
\end{aligned}
$$

Se $g(\varphi,z) = g_0$ *constante,*

$$
\begin{aligned}
u_2(\rho,\varphi,z) &= \sum_j \frac{W_0(\lambda_j\rho)}{W_0(\lambda_j b)} sen\,\lambda_j z \frac{2}{L}\int_0^L sen\,(j\pi y/L)\,dy\, g_0\,, \\
&= \frac{4g_0}{\pi}\sum_j \frac{W_0((2j-1)\pi\rho/L)}{W_0((2j-1)\pi b/L)} \frac{sen\,[(2j-1)\pi z/L]}{2j-1}\,.
\end{aligned}
$$

Se $g_0 = 0$ *temos* $u_2 = 0$.

(c) Cálculo de u_3.

A solução u_3 *é,*

$$
\begin{aligned}
u_3(\rho,\varphi,z) &= \sum_{\lambda\mu} [a_{1\lambda} J_\mu(\lambda\rho) + a_{2\lambda} Y_\mu(\lambda\rho)] \times \\
&\times [b_{1\mu}\cos\mu\varphi + b_{2\mu} sen\,\mu\varphi] \times \\
&\times [c_{1\lambda} cosh\lambda(L-z) + c_{2\lambda} senh\lambda(L-z)]\,,
\end{aligned}
$$

com as condições de contorno,

$$
\begin{aligned}
u_3(a,\varphi,z) &= 0 = \sum_{\lambda\mu}[a_{1\lambda}J_\mu(\lambda a) + a_{2\lambda}Y_\mu(\lambda a)] \times \\
&\times[b_{1\mu}\cos\mu\varphi + b_{2\mu} sen\,\mu\varphi] \times \\
&\times[c_{1\lambda} cosh\lambda(L-z) + c_{2\lambda} senh\lambda(L-z)]\,, \\
u_3(b,\varphi,z) &= 0 = \sum_{\lambda\mu}[a_{1\lambda}J_\mu(\lambda b) + a_{2\lambda}Y_\mu(\lambda b)] \times \\
&\times[b_{1\mu}\cos\mu\varphi + b_{2\mu} sen\,\mu\varphi] \times \\
&\times[c_{1\lambda} cosh\lambda(L-z) + c_{2\lambda} senh\lambda(L-z)]\,, \\
u_3(\rho,\varphi,0) &= h(\rho,\varphi) = \sum_{\lambda\mu}[a_{1\lambda}J_\mu(\lambda\rho) + a_{2\lambda}Y_\mu(\lambda\rho)] \times \\
&\times[b_{1\mu}\cos\mu\varphi + b_{2\mu} sen\,\mu\varphi] \times \\
&\times[c_{1\lambda} cosh\lambda L + c_{2\lambda} senh\lambda L]\,, \\
u_3(\rho,\varphi,L) &= 0 = \sum_{\lambda\mu}[a_{1\lambda}J_\mu(\lambda\rho) + a_{2\lambda}Y_\mu(\lambda\rho)] \times \\
&\times[b_{1\mu}\cos\mu\varphi + b_{2\mu} sen\,\mu\varphi]c_{1\lambda}\,.
\end{aligned}
$$

Satisfazemos as condições acima escolhendo,

$$
\begin{aligned}
&a_{1\lambda}J_\mu(\lambda a) + a_{2\lambda}Y_\mu(\lambda a) = 0\,, \\
&a_{1\lambda}J_\mu(\lambda b) + a_{2\lambda}Y_\mu(\lambda b) = 0\,, \\
&c_{1\lambda} = 0\,.
\end{aligned}
$$

temos,

$$
\begin{aligned}
&J_\mu(\lambda_j a)Y_\mu(\lambda_j b) - J_\mu(\lambda_j b)Y_\mu(\lambda_j a) = 0\,, \quad j = 1,2,\ldots \\
&a_{2j} = -a_{1j}\frac{J_\mu(\lambda_j a)}{Y_\mu(\lambda_j a)} = -a_{1j}\frac{J_\mu(\lambda_j b)}{Y_\mu(\lambda_j b)}\,.
\end{aligned}
$$

A equação que determina λ_j é então,

$$
U_\mu(\lambda_{\mu j} b) = 0\,, \quad \mu = 0,1,2,\ldots, \quad j = 1,2,\ldots
$$

Com isso a solução fica, fazendo $a_{1j} = c_{2j} = 1$,

$$
u_3(\rho,\varphi,z) = \sum_{\mu j}\frac{U_\mu(\lambda_{\mu j}\rho)}{Y_\mu(\lambda_{\mu j} a)}[b_{1\mu}\cos\mu\varphi + b_{2\mu} sen\,\mu\varphi] senh\lambda_{\mu j}(L-z)\,,
$$

e a condição para $h(\rho,\varphi)$ *fica,*

$$h(\rho,\varphi) \;=\; \sum_{\mu j}\frac{U_\mu(\lambda_{\mu j}\rho)}{Y_\mu(\lambda_{\mu j}a)}[b_{1\mu}\cos\mu\varphi + b_{2\mu} sen\,\mu\varphi] senh\lambda_{\mu j}L\,.$$

A expressão para h é uma série de Fourier, logo,

$$\sum_j \frac{U_0(\lambda_{0j}\rho)}{Y_0(\lambda_{0j}a)} b_{10} senh\lambda_{0j}L = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\rho,y)dy\,,$$
$$\sum_j \frac{U_\mu(\lambda_{\mu j}\rho)}{Y_\mu(\lambda_{\mu j}a)} b_{1\mu} senh\lambda_{\mu j}L = \frac{1}{\pi}\int_{-\pi}^{\pi} h(\rho,y)\cos\mu y\,dy\,,$$
$$\sum_j \frac{U_\mu(\lambda_{\mu j}\rho)}{Y_\mu(\lambda_{\mu j}a)} b_{2\mu} senh\lambda_{\mu j}L = \frac{1}{\pi}\int_{-\pi}^{\pi} h(\rho,y) sen\,\mu y\,dy\,,\quad \mu = 1,2,\ldots$$

As expressões acima são séries de funções de Bessel, assim,

$$\frac{b_{10}}{Y_0(\lambda_{0j}a)} senh\lambda_{0j}L = \frac{1}{\int_a^b v[U_0(\lambda_{0j}v)]^2 dv}\int_a^b xU_0(\lambda_{0j}x)dx\frac{1}{2\pi}\int_{-\pi}^{\pi} h(x,y)dy\,,$$
$$\frac{b_{1\mu}}{Y_\mu(\lambda_{\mu j}a)} senh\lambda_{\mu j}L = \frac{1}{\int_a^b v[U_\mu(\lambda_{\mu j}v)]^2 dv}\int_a^b xU_\mu(\lambda_{\mu j}x)dx\frac{1}{\pi}\int_{-\pi}^{\pi} h(x,y)\cos\mu y\,dy\,,$$
$$\frac{b_{2\mu}}{Y_\mu(\lambda_{\mu j}a)} senh\lambda_{\mu j}L = \frac{1}{\int_a^b v[U_\mu(\lambda_{\mu j}v)]^2 dv}\int_a^b xU_\mu(\lambda_{\mu j}x)dx\frac{1}{\pi}\int_{-\pi}^{\pi} h(x,y) sen\,\mu y\,dy\,,$$
$$\mu = 1,2,\ldots$$

ou,

$$b_{10} = \frac{1}{senh\lambda_{0j}L}\frac{Y_0(\lambda_{0j}a)}{\displaystyle\int_a^b v[U_0(\lambda_{0j}v)]^2 dv}\int_a^b xU_0(\lambda_{0j}x)dx\frac{1}{2\pi}\int_{-\pi}^{\pi} h(x,y)dy\,,$$

$$b_{1\mu} = \frac{1}{senh\lambda_{\mu j}L}\frac{Y_\mu(\lambda_{\mu j}a)}{\displaystyle\int_a^b v[U_\mu(\lambda_{\mu j}v)]^2 dv}\int_a^b xU_\mu(\lambda_{\mu j}x)dx\frac{1}{\pi}\int_{-\pi}^{\pi} h(x,y)\cos\mu y\,dy\,,$$

$$b_{2\mu} = \frac{1}{senh\lambda_{\mu j}L}\frac{Y_\mu(\lambda_{\mu j}a)}{\displaystyle\int_a^b v[U_\mu(\lambda_{\mu j}v)]^2 dv}\int_a^b xU_\mu(\lambda_{\mu j}x)dx\frac{1}{\pi}\int_{-\pi}^{\pi} h(x,y) sen\,\mu y\,dy\,,$$
$$\mu = 1,2,\ldots$$

A expansão em série para $h(\rho,\varphi)$ é assim,

$$\begin{aligned} h(\rho,\varphi) &= \sum_j \frac{U_0(\lambda_{0j}\rho)}{\displaystyle\int_a^b v[U_0(\lambda_{0j}v)]^2dv} \int_a^b xU_0(\lambda_{0j}x)dx\frac{1}{2\pi}\int_{-\pi}^{\pi} h(x,y)dy \\ &+\sum_{\mu j} \cos\mu\varphi \times \\ &\times \frac{U_\mu(\lambda_{\mu j}\rho)}{\displaystyle\int_a^b v[U_\mu(\lambda_{\mu j}v)]^2dv} \int_a^b xU_\mu(\lambda_{\mu j}x)dx\frac{1}{\pi}\int_{-\pi}^{\pi} h(x,y)\cos\mu y\, dy \\ &+\sum_{\mu j} sen\,\mu\varphi \times \\ &\times \frac{U_\mu(\lambda_{\mu j}\rho)}{\displaystyle\int_a^b v[U_\mu(\lambda_{\mu j}v)]^2dv} \int_a^b xU_\mu(\lambda_{\mu j}x)dx\frac{1}{\pi}\int_{-\pi}^{\pi} h(x,y) sen\,\mu y\, dy\,. \end{aligned}$$

Se $h(\rho,\varphi) = h_0$ constante,

$$h_0 = h_0 \sum_j \frac{U_0(\lambda_{0j}\rho)}{\displaystyle\int_a^b v[U_0(\lambda_{0j}v)]^2dv} \int_a^b xU_0(\lambda_{0j}x)dx = h_0\,,$$

como esperado.

A solução u_3 é, então,

$$
\begin{aligned}
u_3(\rho,\varphi,z) &= \sum_j \frac{U_0(\lambda_{0j}\rho)}{\displaystyle\int_a^b v[U_0(\lambda_{0j}v)]^2 dv} \frac{senh\lambda_{0j}(L-z)}{senh\lambda_{0j}L} \times \\
&\quad \times \int_a^b xU_0(\lambda_{0j}x)dx \frac{1}{2\pi} \int_{-\pi}^{\pi} h(x,y)dy \\
&\quad + \sum_{\mu j} \frac{U_\mu(\lambda_{\mu j}\rho)}{\displaystyle\int_a^b v[U_\mu(\lambda_{\mu j}v)]^2 dv} \frac{senh\lambda_{\mu j}(L-z)}{senh\lambda_{\mu j}L} \cos\mu\varphi \times \\
&\quad \times \int_a^b xU_\mu(\lambda_{\mu j}x)dx \frac{1}{\pi} \int_{-\pi}^{\pi} h(x,y)\cos\mu y\, dy \\
&\quad + \sum_{\mu j} \frac{U_\mu(\lambda_{\mu j}\rho)}{\displaystyle\int_a^b v[U_\mu(\lambda_{\mu j}v)]^2 dv} \frac{senh\lambda_{\mu j}(L-z)}{senh\lambda_{\mu j}L} sen\,\mu\varphi \times \\
&\quad \times \int_a^b xU_\mu(\lambda_{\mu j}x)dx \frac{1}{\pi} \int_{-\pi}^{\pi} h(x,y) sen\,\mu y\, dy\,.
\end{aligned}
$$

Se $h(\rho,\varphi) = h_0$ constante,

$$
\begin{aligned}
u_3(\rho,\varphi,z) &= \sum_j \frac{U_0(\lambda_{0j}\rho)}{\displaystyle\int_a^b v[U_0(\lambda_{0j}v)]^2 dv} \frac{senh\lambda_{0j}(L-z)}{senh\lambda_{0j}L} \int_a^b xU_0(\lambda_{0j}x)dx\, h_0 \\
&= h_0 \sum_j \frac{U_0(\lambda_{0j}\rho)}{\displaystyle\int_a^b v[U_0(\lambda_{0j}v)]^2 dv} \frac{senh\lambda_{0j}(L-z)}{senh\lambda_{0j}L} \int_a^b xU_0(\lambda_{0j}x)dx\,.
\end{aligned}
$$

Se $h_0 = 0$ temos $u_3 = 0$.

(d) Cálculo de u_4.

Escrevemos agora,

$$
\begin{aligned}
u_4(\rho,\varphi,z) &= \sum_{\lambda\mu} [a_{1\lambda}J_\mu(\lambda\rho) + a_{2\lambda}Y_\mu(\lambda\rho)] \times \\
&\quad \times [b_{1\mu}\cos\mu\varphi + b_{2\mu} sen\,\mu\varphi][c_{1\lambda} cosh\lambda z + c_{2\lambda} senh\lambda z]\,.
\end{aligned}
$$

As condições de contorno são,

$$\begin{aligned}
u_4(a,\varphi,z) &= 0 = \sum_{\lambda\mu}[a_{1\lambda}J_\mu(\lambda a)+a_{2\lambda}Y_\mu(\lambda a)]\times\\
&\times[b_{1\mu}\cos\mu\varphi+b_{2\mu}sen\,\mu\varphi][c_{1\lambda}cosh\lambda z+c_{2\lambda}senh\lambda z]\,,\\
u_4(b,\varphi,z) &= 0 = \sum_{\lambda\mu}[a_{1\lambda}J_\mu(\lambda b)+a_{2\lambda}Y_\mu(\lambda b)]\times\\
&\times[b_{1\mu}\cos\mu\varphi+b_{2\mu}sen\,\mu\varphi][c_{1\lambda}cosh\lambda z+c_{2\lambda}senh\lambda z]\,,\\
u_4(\rho,\varphi,0) &= 0 = \sum_{\lambda\mu}[a_{1\lambda}J_\mu(\lambda\rho)+a_{2\lambda}Y_\mu(\lambda\rho)]\times\\
&\times[b_{1\mu}\cos\mu\varphi+b_{2\mu}sen\,\mu\varphi]c_{1\lambda}\,,\\
u_4(\rho,\varphi,L) &= v(\rho,\varphi) = \sum_{\lambda\mu}[a_{1\lambda}J_\mu(\lambda\rho)+a_{2\lambda}Y_\mu(\lambda\rho)]\times\\
&\times[b_{1\mu}\cos\mu\varphi+b_{2\mu}sen\,\mu\varphi][c_{1\lambda}cosh\lambda L+c_{2\lambda}senh\lambda L]\,.
\end{aligned}$$

Temos agora,

$$\begin{aligned}
&a_{1\lambda\mu}J_\mu(\lambda a)+a_{2\lambda\mu}Y_\mu(\lambda a)=0\,,\\
&a_{1\lambda\mu}J_\mu(\lambda b)+a_{2\lambda\mu}Y_\mu(\lambda b)=0\,,\\
&c_{1\lambda}=0\,,
\end{aligned}$$

portanto,

$$\begin{aligned}
&a_{2\lambda\mu}=-a_{1\lambda\mu}\frac{J_\mu(\lambda a)}{Y_\mu(\lambda a)}\,,\\
&J_\mu(\lambda_j a)Y_\mu(\lambda_j b)-Y_\mu(\lambda_j a)J_\mu(\lambda_j b)=0\,,\quad j=1,2,\ldots
\end{aligned}$$

ou,

$$U_\mu(\lambda_{\mu j}b)=0\,,\quad \mu=0,1,2,\ldots,\quad j=1,2,\ldots$$

A solução u_4 é então, fazendo $a_{1j}=c_{2j}=1$,

$$\begin{aligned}
u_4(\rho,\varphi,z) &= \sum_{j\mu}[J_\mu(\lambda_{\mu j}\rho)Y_\mu(\lambda_{\mu j}a)-J_\mu(\lambda_{\mu j}a)Y_\mu(\lambda_{\mu j}\rho)]\frac{1}{Y_\mu(\lambda_{\mu j}a)}\times\\
&\times[b_{1\mu}\cos\mu\varphi+b_{2\mu}sen\,\mu\varphi]senh\lambda_{\mu j}z\,,
\end{aligned}$$

ou,

$$u_4(\rho,\varphi,z) \;=\; \sum_\mu \sum_j \frac{U_\mu(\lambda_{\mu j}\rho)}{Y_\mu(\lambda_{\mu j}a)} senh\lambda_{\mu j} z[b_{1\mu}\cos\mu\varphi + b_{2\mu} sen\,\mu\varphi]\,.$$

A condição de contorno para v nos dá,

$$v(\rho,\varphi) \;=\; \sum_\mu \sum_j \frac{U_\mu(\lambda_{\mu j}\rho)}{Y_\mu(\lambda_{\mu j}a)} senh\lambda_{\mu j} L[b_{1\mu}\cos\mu\varphi + b_{2\mu} sen\,\mu\varphi]\,,$$

que é uma série de Fourier, portanto,

$$\sum_j \frac{U_0(\lambda_{0j}\rho)}{Y_0(\lambda_{0j}a)} senh\lambda_{0j} L\, b_{10} = \frac{1}{2\pi}\int_{-\pi}^{\pi} v(\rho,y)dy\,,$$

$$\sum_j \frac{U_\mu(\lambda_{\mu j}\rho)}{Y_\mu(\lambda_{\mu j}a)} senh\lambda_{\mu j} L\, b_{1\mu} = \frac{1}{\pi}\int_{-\pi}^{\pi} v(\rho,y)\cos\mu y\, dy\,,$$

$$\sum_j \frac{U_\mu(\lambda_{\mu j}\rho)}{Y_\mu(\lambda_{\mu j}a)} senh\lambda_{\mu j} L\, b_{2\mu} = \frac{1}{\pi}\int_{-\pi}^{\pi} v(\rho,y) sen\,\mu y\, dy\,,$$

$$\mu = 1,2,\ldots$$

As expressões acima são expansões em série de funções de Bessel, logo,

$$\frac{1}{Y_0(\lambda_{0j}a)} senh\lambda_{0j} L b_{10} = \frac{1}{\displaystyle\int_a^b v[U_0(\lambda_{0j}v)]^2 dv}\int_a^b xU_0(\lambda_{0j}x)dx \frac{1}{2\pi}\int_{-\pi}^{\pi} v(x,y)dy\,,$$

$$\frac{1}{Y_\mu(\lambda_{\mu j}a)} senh\lambda_{\mu j} L b_{1\mu} = \frac{1}{\displaystyle\int_a^b v[U_\mu(\lambda_{\mu j}v)]^2 dv}\int_a^b xU_\mu(\lambda_{\mu j}x)dx \frac{1}{\pi}\int_{-\pi}^{\pi} v(x,y)\cos\mu y\, dy\,,$$

$$\frac{1}{Y_\mu(\lambda_{\mu j}a)} senh\lambda_{\mu j} L b_{2\mu} = \frac{1}{\displaystyle\int_a^b v[U_\mu(\lambda_{\mu j}v)]^2 dv}\int_a^b xU_\mu(\lambda_{\mu j}x)dx \frac{1}{\pi}\int_{-\pi}^{\pi} v(x,y) sen\,\mu y\, dy\,,$$

$$\mu = 1,2,\ldots$$

ou,

$$b_{10} = \frac{Y_0(\lambda_{0j}a)}{senh\lambda_{0j}L}\frac{1}{\displaystyle\int_a^b w[U_0(\lambda_{0j}w)]^2dw}\int_a^b xU_0(\lambda_{0j}x)dx\frac{1}{2\pi}\int_{-\pi}^{\pi} v(x,y)dy\,,$$

$$b_{1\mu} = \frac{Y_\mu(\lambda_{\mu j}a)}{senh\lambda_{\mu j}L}\frac{1}{\displaystyle\int_a^b w[U_\mu(\lambda_{\mu j}w)]^2dw}\int_a^b xU_\mu(\lambda_{\mu j}x)dx\frac{1}{\pi}\int_{-\pi}^{\pi} v(x,y)\cos\mu y\,dy\,,$$

$$b_{2\mu} = \frac{Y_\mu(\lambda_{\mu j}a)}{senh\lambda_{\mu j}L}\frac{1}{\displaystyle\int_a^b w[U_\mu(\lambda_{\mu j}w)]^2dw}\int_a^b xU_\mu(\lambda_{\mu j}x)dx\frac{1}{\pi}\int_{-\pi}^{\pi} v(x,y)sen\,\mu y\,dy\,.$$

A série para $v(\rho,\varphi)$ é assim,

$$\begin{aligned} v(\rho,\varphi) &= \sum_j \frac{U_0(\lambda_{0j}\rho)}{\displaystyle\int_a^b w[U_0(\lambda_{0j}w)]^2dw}\int_a^b xU_0(\lambda_{0j}x)dx\frac{1}{2\pi}\int_{-\pi}^{\pi} v(x,y)dy \\ &\quad+\sum_{\mu j}\frac{U_\mu(\lambda_{\mu j}\rho)}{\displaystyle\int_a^b w[U_\mu(\lambda_{\mu j}w)]^2dw}\cos\mu\varphi\,\times \\ &\quad\times\int_a^b xU_\mu(\lambda_{\mu j}x)dx\frac{1}{\pi}\int_{-\pi}^{\pi} v(x,y)\cos\mu y\,dy \\ &\quad+\sum_{\mu j}\frac{U_\mu(\lambda_{\mu j}\rho)}{\displaystyle\int_a^b w[U_\mu(\lambda_{\mu j}w)]^2dw}sen\,\mu\varphi\,\times \\ &\quad\times\int_a^b xU_\mu(\lambda_{\mu j}x)dx\frac{1}{\pi}\int_{-\pi}^{\pi} v(x,y)sen\,\mu y\,dy\,. \end{aligned}$$

Se $v(\rho,\varphi) = v_0$ constante,

$$\begin{aligned} v_0 &= \sum_j \frac{U_0(\lambda_{0j}\rho)}{\displaystyle\int_a^b w[U_0(\lambda_{0j}w)]^2dw}\int_a^b xU_0(\lambda_{0j}x)dx\,v_0\,, \\ &= v_0\sum_j \frac{U_0(\lambda_{0j}\rho)}{\displaystyle\int_a^b w[U_0(\lambda_{0j}w)]^2dw}\int_a^b xU_0(\lambda_{0j}x)dx = v_0\,, \end{aligned}$$

como esperado.

A solução u_4 é então,

$$\begin{aligned} u_4(\rho,\varphi,z) &= \sum_j \frac{U_0(\lambda_{0j}\rho)}{\displaystyle\int_a^b w[U_0(\lambda_{0j}w)]^2dw}\frac{senh\lambda_{0j}z}{senh\lambda_{0j}L}\times \\ &\times \int_a^b xU_0(\lambda_{0j}x)dx\frac{1}{2\pi}\int_{-\pi}^{\pi}v(x,y)dy \\ &+\sum_{\mu j}\frac{U_\mu(\lambda_{\mu j}\rho)}{\displaystyle\int_a^b w[U_\mu(\lambda_{\mu j}w)]^2dw}\frac{senh\lambda_{\mu j}z}{senh\lambda_{\mu j}L}\cos\mu\varphi\times \\ &\times \int_a^b xU_\mu(\lambda_{\mu j}x)dx\frac{1}{\pi}\int_{-\pi}^{\pi}v(x,y)\cos\mu y\,dy \\ &+\sum_{\mu j}\frac{U_\mu(\lambda_{\mu j}\rho)}{\displaystyle\int_a^b w[U_\mu(\lambda_{\mu j}w)]^2dw}\frac{senh\lambda_{\mu j}z}{senh\lambda_{\mu j}L}sen\,\mu\varphi\times \\ &\times \int_a^b xU_\mu(\lambda_{\mu j}x)dx\frac{1}{\pi}\int_{-\pi}^{\pi}v(x,y)sen\,\mu y\,dy\,. \end{aligned}$$

Se $v(\rho,\varphi)=v_0$ constante,

$$\begin{aligned} u_4(\rho,\varphi,z) &= \sum_j \frac{U_0(\lambda_{0j}\rho)}{\displaystyle\int_a^b w[U_0(\lambda_{0j}w)]^2dw}\frac{senh\lambda_{0j}z}{senh\lambda_{0j}L}\int_a^b xU_0(\lambda_{0j}x)dx\,v_0\,, \\ &= v_0\sum_j \frac{U_0(\lambda_{0j}\rho)}{\displaystyle\int_a^b w[U_0(\lambda_{0j}w)]^2dw}\frac{senh\lambda_{0j}z}{senh\lambda_{0j}L}\int_a^b xU_0(\lambda_{0j}x)dx\,. \end{aligned}$$

Se $v_0=0$ temos $u_4=0$.

(e) Se as funções nas condições de contorno são todas constantes,

$$
\begin{aligned}
u(\rho,\varphi,z) &= \frac{4f_0}{\pi}\sum_j \frac{V_0((2j-1)\pi\rho/L)}{V_0((2j-1)\pi a/L)}\frac{sen\,[(2j-1)\pi z/L]}{2j-1} \\
&+\frac{4g_0}{\pi}\sum_j \frac{W_0((2j-1)\pi\rho/L)}{W_0((2j-1)\pi b/L)}\frac{sen\,[(2j-1)\pi z/L]}{2j-1} \\
&+h_0\sum_j \frac{U_0(\lambda_{0j}\rho)}{\int_a^b w[U_0(\lambda_{0j}w)]^2dw}\frac{senh\lambda_{0j}(L-z)}{senh\lambda_{0j}L}\int_a^b xU_0(\lambda_{0j}x)dx \\
&+v_0\sum_j \frac{U_0(\lambda_{0j}\rho)}{\int_a^b w[U_0(\lambda_{0j}w)]^2dw}\frac{senh\lambda_{0j}z}{senh\lambda_{0j}L}\int_a^b xU_0(\lambda_{0j}x)dx\,.
\end{aligned}
$$

3. Considere um cilindro infinito de raio a. Calcule a solução $u(\rho,\varphi,z)$ da equação de Laplace com a condição de contorno,

$$u(a,\varphi,z) = f(\varphi,z)\,.$$

4. Considere um cilindro infinito de raio a. Calcule a solução $u(\rho,\varphi,z)$ da equação de Laplace com a condição de contorno,

$$\frac{\partial u(a,\varphi,z)}{\partial\rho} = f(\varphi,z)\,.$$

5. Considere uma casca cilíndrica infinita de raio interno a e raio externo b. Calcule a solução $u(\rho,\varphi,z)$ da equação de Laplace com as condições de contorno,

$$
\begin{aligned}
u(a,\varphi,z) &= f(\varphi,z)\,,\\
u(b,\varphi,z) &= g(\varphi,z)\,.
\end{aligned}
$$

6. Considere um cilindro em $0<\rho<a$, $0<z<L$ com radiação obedecendo a lei de Newton do resfriamento em $z=0$. Calcule a temperatura no estado estacionário (Spiegel [7], probl. 6.98).

7. Encontre a solução $u(\rho,\varphi,z)$ para as equações:

(a)

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2u}{\partial\varphi^2}+\frac{\partial^2u}{\partial z^2} = -f(\rho,\varphi,z)\,;$$

(b)

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{du}{d\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2u}{\partial\varphi^2}+\frac{\partial^2u}{\partial z^2} = +\alpha^2\,;$$

com α constante,

(c)

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{du}{d\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2}+\frac{\partial^2 u}{\partial z^2}=-\alpha^2\,;$$

com α constante,

(d)

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{du}{d\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2}+\frac{\partial^2 u}{\partial z^2}=+\alpha^2 u\,;$$

com α constante,

(e)

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{du}{d\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2}+\frac{\partial^2 u}{\partial z^2}=-\alpha^2 u\,;$$

com α constante e,

$$\begin{aligned}&0\le\rho\le a\,,\quad 0\le\varphi\le 2\pi\,,0\le z\le L\,,\\&u(a,\varphi,z)=\mu(\varphi,z)\,,\quad u(\rho,\varphi,0)=\nu_1(\rho,\varphi)\,,\\&u(\rho,\varphi,L)=\nu_2(\rho,\varphi)\,.\end{aligned}$$

8. Encontre a solução $u(\rho,\varphi,z)$ para as equações:

(a)

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{du}{d\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2}+\frac{\partial^2 u}{\partial z^2}=-f(\rho,\varphi,z)\,;$$

(b)

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{du}{d\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2}+\frac{\partial^2 u}{\partial z^2}=+\alpha^2\,;$$

com α constante,

(c)

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{du}{d\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2}+\frac{\partial^2 u}{\partial z^2}=-\alpha^2\,;$$

com α constante,

(d)

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{du}{d\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2}+\frac{\partial^2 u}{\partial z^2}=+\alpha^2 u\,;$$

com α constante,

(e)

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{du}{d\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2}+\frac{\partial^2 u}{\partial z^2}=-\alpha^2 u\,;$$

com α constante e,

$$a \le \rho \le b\,, \quad 0 \le \varphi \le 2\pi\,, \quad 0 \le z \le L\,,$$
$$u(a,\varphi,z) = \mu_1(\varphi,z)\,, \quad u(b,\varphi,z) = \mu_2(\varphi,z)\,,$$
$$u(\rho,\varphi,0) = \nu_1(\rho,\varphi)\,, \quad u(\rho,\varphi,L) = \nu_2(\rho,\varphi)\,.$$

9 Apêndice

1. Séries de Fourier.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \mathrm{sen}\frac{n\pi x}{L}\right)\,,$$
$$\frac{a_0}{2} = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx\,,$$
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx\,,$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\mathrm{sen}\frac{n\pi x}{L}\,dx\,, \quad n = 1, 2, \ldots$$

2. Série de Fourier de senos.

$$f(z) = \sum_{j=1} b_j \mathrm{sen}\,(j\pi z/L)\,,$$
$$b_j = \frac{2}{L}\int_0^L f(x)\mathrm{sen}\,(j\pi x/L)\,dx\,.$$

3. Série de Fourier de cosenos.

$$f(z) = \frac{a_0}{2} + \sum_{j=1} a_j \cos\,(j\pi z/L)\,,$$
$$a_j = \frac{2}{L}\int_0^L f(x)\cos\,(j\pi x/L)\,dx\,.$$

4. Polinômios de Legendre ($x = \cos\theta$).

$$
\begin{aligned}
P_0(x) &= 1\\
P_1(x) &= x\\
P_2(x) &= \frac{1}{2}(3x^2-1)\\
P_3(x) &= \frac{1}{2}(5x^3-3x)\\
P_4(x) &= \frac{1}{8}(35x^4-30x^2+3)\\
P_5(x) &= \frac{1}{8}(63x^5-70x^3+15x)\\
P_6(x) &= \frac{1}{16}(231x^6-315x^4+105x^2-5)\\
P_7(x) &= \frac{1}{16}(497x^7-693x^5+315x^3-35x)\\
P_8(x) &= \frac{1}{128}(6435x^8-12012x^6+6930x^4-1260x^2+35)
\end{aligned}
$$

5. Relação de ortogonalidade.

$$
\int_{-1}^{+1} P_n(x)P_k(x)dx = \begin{cases} 0\,, & n \neq k\,,\\ \dfrac{2}{2n+1}\,, & n = k\,, \end{cases}
$$

6. Série de polinômios de Legendre.

$$
\begin{aligned}
f(\theta) &= \sum_{k=0}^{\infty} C_k P_k(\cos\theta)\,,\\
C_k &= \frac{2k+1}{2}\int_0^{\pi} f(\theta)P_k(\cos\theta)\mathrm{sen}\,\theta\,d\theta\,.
\end{aligned}
$$

7. Funções associadas de Legendre.

$$
\begin{aligned}
P_1^1(x) &= (1-x^2)^{1/2} = \operatorname{sen}\theta \\
P_2^1(x) &= 3x(1-x^2)^{1/2} = 3\operatorname{sen}\theta\cos\theta \\
P_2^2(x) &= 3(1-x^2) = 3\operatorname{sen}^2\theta \\
P_3^1(x) &= \frac{3}{2}(5x^2-1)(1-x^2)^{1/2} = \frac{3}{2}(5\cos^2\theta-1)\operatorname{sen}\theta \\
P_3^2(x) &= 15x(1-x^2) = 15\operatorname{sen}^2\theta\cos\theta \\
P_3^3(x) &= 15(1-x^2)^{3/2} = 15\operatorname{sen}^3\theta \\
P_4^1(x) &= \frac{5}{2}(7x^3-3x)(1-x^2)^{1/2} = \frac{5}{2}(7\cos^3\theta-3\cos\theta)\operatorname{sen}\theta \\
P_4^2(x) &= \frac{15}{2}(7x^2-1)(1-x^2) = \frac{15}{2}(7\cos^2\theta-1)\operatorname{sen}^2\theta \\
P_4^3(x) &= 105x(1-x^2)^{3/2} = 105\operatorname{sen}^3\theta\cos\theta \\
P_4^4(x) &= 105(1-x^2)^2 = 105\operatorname{sen}^4\theta
\end{aligned}
$$

8. Relação de ortogonalidade.

$$
\int_{-1}^{+1} P_n^m(x)P_k^m(x)dx = \begin{cases} 0\,, & n \neq k\,, \\ \dfrac{2}{2n+1}\dfrac{(n+m)!}{(n-m)!}\,, & n = k\,, \end{cases}
$$

9. Série de funções associadas de Legendre do primeiro tipo.

$$
\begin{aligned}
f(\theta) &= \sum_{k=0}^{\infty} D_k P_k^m(\cos\theta)\,, \\
D_k &= \frac{(2k+1)(k-m)!}{2(k+m)!}\int_0^{\pi} f(\theta)P_k^m(\cos\theta)\operatorname{sen}\theta\, d\theta\,,
\end{aligned}
$$

10.

$$
\begin{aligned}
\int_0^L \operatorname{sen}(j\pi x/L)\, dx &= \begin{cases} 0\,, & j \text{ par}\,, \\ \dfrac{2L}{j\pi}\,, & j \text{ ímpar}\,. \end{cases} \\
&= \frac{2L}{(2j-1)\pi}\,, j = 1, 2, 3, \ldots
\end{aligned}
$$

11.

$$
\int_0^L x\operatorname{sen}(j\pi x/L)\, dx = \frac{L^2}{j\pi}(-1)^{j+1}\,, j = 1, 2, 3, \ldots
$$

12.

$$\int_0^{+L} \cos(k\pi x/2L)dx = \frac{2L}{k\pi}\text{sen}\,(k\pi/2)\,.$$

13.

$$\int_0^{L} \cos(i\pi x/L)\cos(j\pi x/L)dx = \begin{cases} 0\,, i \neq j\,, \\ \dfrac{L}{2}\,, i = j\,. \end{cases}$$

14.

$$\int_0^{+L} \cos[(2j-1)\pi z/2L]\cos[(2k-1)\pi z/2L]dz = \begin{cases} 0\,, j \neq k\,, \\ \dfrac{L}{2}\,, j = k\,. \end{cases}$$

15.

$$\delta(z-x) = \frac{2}{L}\sum_{j=1}\text{sen}\,(j\pi z/L)\text{sen}\,(j\pi x/L)\,,$$
$$0 \leq x, z \leq L\,.$$

16.

$$\delta(z-x) = \frac{2}{L}\sum_{j=1}\text{sen}\,[(2j-1)\pi z/2L]\text{sen}\,((2j-1)\pi x/2L)\,,$$
$$0 \leq x, z \leq L\,.$$

17.

$$\delta(z-x) = \frac{2}{L}\sum_{j=1}\cos[(2j-1)\pi z/2L]\cos[(2j-1)\pi x/2L]\,,$$
$$0 \leq x, z \leq L\,.$$

18.

$$1 = \frac{4}{\pi}\sum_{j=1}\frac{\text{sen}\,[(2j-1)\pi z/2L]}{2j-1}\,,\quad 0 \leq x \leq L\,.$$

19.

$$1 = \frac{4}{\pi} \sum_{j=1} \frac{1}{2j-1} \cos[(2j-1)\pi z/2L] \text{sen}\,[(2j-1)\pi/2]\,, \quad 0 \leq x \leq L\,.$$

20. Funções de Bessel.

$$x J_n'(x) = n J_n(x) - x J_{n+1}(x)\,,$$

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x)\,,$$

$$\int x J_0(x) dx = x J_1(x)\,,$$

$$\int x^n J_{n-1}(x) dx = x^n J_n(x)\,,$$

$$\int_0^a \rho' J_0(\lambda_i \rho') d\rho' = \frac{a}{\lambda_i} J_1(\lambda_i a)\,,$$

***

(a)

$$f(x) = \sum_{p=1}^{\infty} A_p J_n(\lambda_p x)\,, \quad 0 < x < a\,,$$

$$J_n(\lambda_p a) = 0\,, \quad p = 1, 2, 3, \ldots\,,$$

$$A_k = \frac{2}{a^2 J_{n+1}^2(\lambda_k a)} \int_0^a x J_n(\lambda_k x) f(x) dx\,.$$

(b)

$$f(x) = \sum_{p=1}^{\infty} A_p J_n(\lambda_p x)\,, \quad 0 < x < a\,,$$

$$J_n'(\lambda_p a) = 0\,, \quad p = 1, 2, 3, \ldots,$$

$$A_k = \frac{2}{a^2(1 - n^2/(\lambda_k a)^2) J_n^2(\lambda_k a)} \int_0^a x J_n(\lambda_k x) f(x) dx\,.$$

Em particular, se temos,

$$J_1(\lambda_p a) = 0\,, \quad p = 1, 2, 3, \ldots,$$

então,

$$f(x) = A_0 + \sum_{p=1}^{\infty} A_p J_0(\lambda_p x)\,, \quad 0 < x < a\,,$$

com,

$$A_0 = \frac{2}{a^2}\int_0^a x f(x) dx\,,$$
$$A_k = \frac{2}{a^2 J_0^2(\lambda_k a)}\int_0^a x J_0(\lambda_k x) f(x) dx\,.$$

(c)

$$f(x) = \sum_{p=1}^{\infty} A_p J_n(\lambda_p x)\,, \quad 0 < x < a\,,$$

$$R J_n(\lambda_p a) + S\lambda_p a J_n'(\lambda_p a) = 0\,, \quad p = 1, 2, 3, \ldots,$$

$$A_k \;=\; \frac{2(\lambda_k a)^2 \displaystyle\int_0^a x J_n(\lambda_k x) f(x) dx}{a^2 J_n^2(\lambda_k a)[R^2/S^2 + (\lambda_k a)^2 - n^2]}\,.$$

(d)

$$1 = \frac{2}{a}\sum_{j=1} \frac{J_0(\lambda_j \rho)}{\lambda_j J_1(\lambda_j a)}\,, \quad J_0(\lambda_j a) = 0\,.$$

(e)

$$\delta(\rho - x) = \frac{2}{a^2}\sum_j \frac{J_0(\lambda_j \rho)}{J_1^2(\lambda_j a)} x J_0(\lambda_j x)\,, \quad J_0(\lambda_j a) = 0\,. \;\; 0 \le \rho \le a\,.$$

(f)

$$U_n(\lambda_j \rho) \equiv J_n(\lambda_j \rho) Y_n(\lambda_j a) - J_n(\lambda_j a) Y_n(\lambda_j \rho)\,,$$

$$U_n(\lambda_j b) = 0\,,$$

$$\int_a^b \rho U_\mu(\lambda_j \rho) U_\mu(\lambda_k \rho) d\rho = 0 \,, \quad j \neq k \,.$$

(g)

$$f(\rho) = \sum_{j=1} A_j U_\mu(\lambda_j \rho) \,,$$

$$A_j = \frac{\displaystyle\int_a^b \rho f(\rho) U_\mu(\lambda_j \rho) d\rho}{\displaystyle\int_a^b \rho U_\mu^2(\lambda_j \rho) d\rho} \,.$$

(h)

$$\delta(\rho - x) = \sum_j \frac{U_0(\lambda_j \rho)}{\displaystyle\int_a^b y [U_0(\lambda_j y)]^2 dy} x U_0(\lambda_j x) \,, \quad a \leq \rho \leq b \,,$$

(i)

$$1 = \sum_j U_0(\lambda_j \rho) \frac{\displaystyle\int_a^b x U_0(\lambda_j x) dx}{\displaystyle\int_a^b y [U_0(\lambda_j y)]^2 dy} \,.$$

(j)

$$V_n(\lambda_j \rho) \equiv I_n(\lambda_j \rho) K_n(\lambda_j b) - I_n(\lambda_j b) K_n(\lambda_j \rho) \,,$$

(k)

$$W_n(\lambda_j \rho) \equiv I_n(\lambda_j \rho) K_n(\lambda_j a) - I_n(\lambda_j a) K_n(\lambda_j \rho) \,,$$

$$Z_n(\lambda_j \rho) \equiv J_n(\lambda_j \rho) Y_n(\lambda_j b) - J_n(\lambda_j b) Y_n(\lambda_j \rho) \,,$$

(l)

$$f(x) = x^2 = \frac{2}{a} \sum_{k=1}^{\infty} \frac{J_0(\lambda_k x)}{J_1(\lambda_k a)} \frac{1}{\lambda_k^3} \left[(\lambda_k a)^2 - 4 \right] \,, \quad J_0(\lambda_k a) = 0 \,.$$

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