11 - A equação da onda em coordenadas esféricas

A equação da onda é,

$$\nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}\,, \tag{1}$$

em que $u = u(r, \theta, \varphi, t)$ e v é uma constante com dimensões de velocidade. É a velocidade de propagação da onda. Separando as coordenadas espaciais do tempo,

$$u(t, r, \theta, \varphi) = T(t)F(r, \theta, \varphi)\,. \tag{2}$$

Substituindo (69) em (68) temos,

$$T\,\nabla^2 F = \frac{1}{v^2}FT''\,,$$

ou,

$$\frac{1}{F}\nabla^2 F = \frac{1}{v^2}\frac{T''}{T} \equiv -\lambda^2\,. \tag{3}$$

A equação para T é então,

$$T'' + \lambda^2 v^2 T = 0\,, \tag{4}$$

com solução,

$$T(t) = Ae^{-i\omega t}\,, \tag{5}$$

em que definimos a frequência angular,

$$\omega = \lambda v\,. \tag{6}$$

Notemos que a constante de separação k tem dimensões de número de onda, ou comprimento^{-1}, sendo v a velocidade de propagação da onda. Notemos que em (5) podemos ter também um termo análogo com $+i\omega t$. Também podemos escrever T na forma,

$$T(t) = a_1 \cos \omega t + a_2 \mathrm{sen}\omega t\,. \tag{7}$$

A equação para F é,

$$\nabla^2 F + \lambda^2 F = 0\,. \tag{8}$$

Em coordenadas esféricas,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial F}{\partial r}\right)+\frac{1}{r^2\text{sen}\theta}\frac{\partial}{\partial\theta}\left(\text{sen}\theta\frac{\partial F}{\partial\theta}\right)+\frac{1}{r^2\text{sen}^2\theta}\frac{\partial^2 F}{\partial\varphi^2}+\lambda^2F=0\,. \qquad (9)$$

Assim, a parte espacial da equação da onda é idêntica à parte espacial da equação do calor. Podemos então usar os resultados anteriores, considerando as condições de contorno adequadas.

1 Problemas

1. Considere uma esfera de raio a. Calcule $u(t,r)$ sendo $u(t,a)=u_a$ e $u(0,r)=f(r)$.

A solução é,

$$u(t,r)=u_0(r)+\sum_j r^{-1/2}J_{1/2}(\lambda_j r)[a_1\cos\lambda_j vt+a_2 sen\,\lambda_j vt]\,.$$

A condição de contorno em $r=a$ nos dá,

$$u(t,a)=u_a=u_0(a)+\sum_j a^{-1/2}J_{1/2}(\lambda_j a)[a_1\cos\lambda_j vt+a_2 sen\,\lambda_j vt]\,.$$

Escolhendo,

$$J_{1/2}(\lambda_j a)=0\,,\quad j=1,2,\ldots$$

determinamos os valores possíveis de λ. Também temos,

$$u_a=u_0(a)\,,$$

e portanto $u_0(r)=u_a$. A condição inicial é,

$$u(0,r)=f(r)=u_a+\sum_j r^{-1/2}J_{1/2}(\lambda_j r)a_1\,,$$

ou,

$$f(r)-u_a=\sum_j r^{-1/2}J_{1/2}(\lambda_j r)a_1\,.$$

Temos uma série de funções de Bessel. Multiplicando por $r^{3/2}J_{1/2}(\lambda_k r)$ e integrando em r,

$$\int_0^a dr\, r^{3/2} J_{1/2}(\lambda_k r)[f(r) - u_a] = \sum_j \int_0^a dr\, r J_{1/2}(\lambda_j r) J_{1/2}(\lambda_k r) a_1\,.$$

Usando a relação de ortogonalidade,

$$\int_0^a r J_{1/2}(\lambda_j r) J_{1/2}(\lambda_k r) dr = \begin{cases} 0\,, & j \neq k\,, \\ \dfrac{a^2}{2}[J'_{1/2}(\lambda_k a)]^2\,, & j = k\,, \end{cases}$$

temos,

$$\int_0^a dr\, r^{3/2} J_{1/2}(\lambda_k r)[f(r) - u_a] = \frac{a^2}{2}[J'_{1/2}(\lambda_k a)]^2 a_1\,.$$

Os coeficientes na solução são portanto,

$$a_1 = \frac{2}{a^2[J'_{1/2}(\lambda_k a)]^2} \int_0^a dr\, r^{3/2} J_{1/2}(\lambda_k r)[f(r) - u_a]\,.$$

Temos então, fazendo $a_2 = 0$,

$$\begin{aligned} u(t,r) = u_a + \sum_j r^{-1/2} J_{1/2}(\lambda_j r) \cos \lambda_j v t \times \\ \frac{2}{a^2[J'_{1/2}(\lambda_j a)]^2} \int_0^a dr\, r^{3/2} J_{1/2}(\lambda_j r)[f(r) - u_a]\,. \end{aligned}$$

A expressão acima é idêntica à obtida para a equação do calor, com exceção do termo dependente do tempo.

2. Considere a região $a \leq r \leq b$. Calcule $u(t,r)$ sendo $u(t,a) = u_a$, $u(t,b) = u_b$ e $u(0,r) = f(r)$.

Escrevemos a solução como,

$$\begin{aligned} u(t,r) = u_0(r) + \sum_j r^{-1/2}[b_1 J_{1/2}(\lambda_j r) + b_2 Y_{1/2}(\lambda_j r)] \times \\ \times [a_1 \cos \lambda_j v t + a_2 sen\, \lambda_j v t]\,. \end{aligned}$$

Em $r = a$ e $r = b$,

$$
\begin{aligned}
u(t,a) = u_a = u_0(a) + \sum_j a^{-1/2}[b_1 J_{1/2}(\lambda_j a) + b_2 Y_{1/2}(\lambda_j a)] \times \\
\times[a_1 \cos\lambda_j vt + a_2 sen\, \lambda_j vt] \\
u(t,b) = u_b = u_0(b) + \sum_j b^{-1/2}[b_1 J_{1/2}(\lambda_j b) + b_2 Y_{1/2}(\lambda_j b)] \times \\
\times[a_1 \cos\lambda_j vt + a_2 sen\, \lambda_j vt]\,.
\end{aligned}
$$

Satisfazemos as condições acima escolhendo,

$$
\begin{aligned}
b_1 J_{1/2}(\lambda_j a) + b_2 Y_{1/2}(\lambda_j a) = 0\,, \\
b_1 J_{1/2}(\lambda_j b) + b_2 Y_{1/2}(\lambda_j b) = 0\,.
\end{aligned}
$$

Portanto,

$$
\begin{aligned}
u_a = u_0(a)\,, \\
u_b = u_0(b)\,.
\end{aligned}
$$

Os valores de λ são assim determinados por,

$$J_{1/2}(\lambda_j a) Y_{1/2}(\lambda_j b) - Y_{1/2}(\lambda_j a) J_{1/2}(\lambda_j b) = 0\,,$$

ou,

$$u_{1/2}(\lambda_j b) = 0\,, \quad j = 1, 2, \ldots$$

com,

$$u_{1/2}(\lambda_j r) \equiv Y_{1/2}(\lambda_j a) J_{1/2}(\lambda_j r) - J_{1/2}(\lambda_j a) Y_{1/2}(\lambda_j r)\,.$$

Podemos escrever também,

$$b_2 = -b_1 \frac{J_{1/2}(\lambda_j a)}{Y_{1/2}(\lambda_j a)}\,,$$

logo a solução fica, fazendo $b_1 = 1$,

$$
\begin{aligned}
u(t,r) = u_0(r) + \sum_j r^{-1/2} u_{1/2}(\lambda_j r) \frac{1}{Y_{1/2}(\lambda_j a)} \times \\
\times[a_1 \cos\lambda_j vt + a_2 sen\, \lambda_j vt]\,.
\end{aligned}
$$

A condição inicial é,

$$u(0,r) = f(r) = u_0(r) + \sum_j r^{-1/2} u_{1/2}(\lambda_j r) \frac{1}{Y_{1/2}(\lambda_j a)} a_1 \,,$$

ou,

$$f(r) - u_0(r) = \sum_j r^{-1/2} u_{1/2}(\lambda_j r) \frac{1}{Y_{1/2}(\lambda_j a)} a_1 \,.$$

Temos agora uma série de funções de Bessel. Multiplicando por $r^{3/2} u_{1/2}(\lambda_k r)$ *e integrando em* r*,*

$$\int_a^b dr\, r^{3/2} u_{1/2}(\lambda_k r)[f(r) - u_0(r)] = \\ = \sum_j \int_a^b dr\, r u_{1/2}(\lambda_j r) u_{1/2}(\lambda_k r) \frac{1}{Y_{1/2}(\lambda_j a)} a_1 \,.$$

Usando a condição de ortogonalidade,

$$\int_a^b r u_{1/2}(\lambda_j r) u_{1/2}(\lambda_k r) dr = 0 \,, \quad j \neq k \,,$$

temos,

$$\int_a^b dr\, r^{3/2} u_{1/2}(\lambda_k r)[f(r) - u_0(r)] = \int_a^b dr\, r u_{1/2}^2(\lambda_k r) \frac{1}{Y_{1/2}(\lambda_k a)} a_1 \,.$$

O coeficiente a_1 *é então,*

$$a_1 = \frac{Y_{1/2}(\lambda_k a)}{\int_a^b dr\, r u_{1/2}^2(\lambda_k r)} \int_a^b dr\, r^{3/2} u_{1/2}(\lambda_k r)[f(r) - u_0(r)] \,,$$

e a solução é, fazendo $a_2 = 0$*,*

$$u(t,r) = u_0(r) + \sum_j r^{-1/2} u_{1/2}(\lambda_j r) \cos \lambda_j v t \, \times \\ \times \frac{1}{\int_a^b dr\, r u_{1/2}^2(\lambda_j r)} \int_a^b dr\, r^{3/2} u_{1/2}(\lambda_j r)[f(r) - u_0(r)] \,.$$

A parte espacial da solução acima concorda com o caso correspondente da equação do calor, como esperado.

3. Considere uma esfera de raio a. Calcule $u(t,r,\theta)$ sendo $u(t,a,\theta) = f(\theta)$ e $u(0,r,\theta) = g(r,\theta)$.

(a) A solução finita é,

$$u(t,r,\theta) = u_0(r,\theta) + \sum_{\lambda n} r^{-1/2} J_{n+1/2}(\lambda r) P_n(\cos\theta) \times$$
$$\times [a_1 \cos \lambda vt + a_2 sen\, \lambda vt] \,.$$

A condição de contorno em $r = a$ é,

$$u(t,a,\theta) = f(\theta) = u_0(a,\theta) + \sum_{\lambda n} r^{-1/2} J_{n+1/2}(\lambda a) P_n(\cos\theta) \times$$
$$\times [a_1 \cos \lambda vt + a_2 sen\, \lambda vt] \,.$$

Escolhendo,

$$J_{n+1/2}(\lambda_{nj} a) = 0 \,, \quad j = 1, 2, \ldots$$

satisfazemos a equação acima e determinamos os valores possíveis de λ. Também temos,

$$f(\theta) = u_0(a,\theta) \,.$$

A condição inicial nos dá,

$$u(0,r,\theta) = g(r,\theta) = u_0(r,\theta) + \sum_{jn} r^{-1/2} J_{n+1/2}(\lambda_{nj} r) P_n(\cos\theta) a_1 \,,$$

ou,

$$g(r,\theta) - u_0(r,\theta) = \sum_{jn} r^{-1/2} J_{n+1/2}(\lambda_{nj} r) P_n(\cos\theta) a_1 \,.$$

A expressão acima é uma série de polinômios de Legendre, logo,

$$\sum_j r^{-1/2} J_{n+1/2}(\lambda_{nj} r) a_1 = \frac{2n+1}{2} \int_0^\pi P_n(\cos\theta) sen\,\theta \, d\theta [g(r,\theta) - u_0(r,\theta)] \,.$$

Temos agora uma série de funções de Bessel. Multiplicando por $r^{3/2} J_{n+1/2}(\lambda_{nk} r)$ e integrando em r,

$$\sum_j \int_0^a dr\, r J_{n+1/2}(\lambda_{nj} r) J_{n+1/2}(\lambda_{nk} r) a_1 =$$
$$= \int_0^a dr\, r^{3/2} J_{n+1/2}(\lambda_{nk} r) \times$$
$$\times \frac{2n+1}{2} \int_0^\pi P_n(\cos\theta) sen\,\theta\, d\theta [g(r,\theta) - u_0(r,\theta)]\,.$$

Usando a condição de ortogonalidade,

$$\int_0^a r J_{l+1/2}(\lambda_{ln} r) J_{l+1/2}(\lambda_{lm} r) dr = \begin{cases} 0\,, & n \neq m\,, \\ \frac{a^2}{2}[J'_{l+1/2}(\lambda_{ln} a)]^2\,, & n = m\,, \end{cases}$$

temos,

$$\frac{a^2}{2}[J'_{n+1/2}(\lambda_{nk} a)]^2 a_1 =$$
$$= \int_0^a dr\, r^{3/2} J_{n+1/2}(\lambda_{nk} r) \times$$
$$\times \frac{2n+1}{2} \int_0^\pi P_n(\cos\theta) sen\,\theta\, d\theta [g(r,\theta) - u_0(r,\theta)]\,,$$

ou,

$$a_1 = \frac{2}{a^2[J'_{n+1/2}(\lambda_{nk} a)]^2} \int_0^a dr\, r^{3/2} J_{n+1/2}(\lambda_{nk} r) \times$$
$$\times \frac{2n+1}{2} \int_0^\pi P_n(\cos\theta) sen\,\theta\, d\theta [g(r,\theta) - u_0(r,\theta)]\,.$$

A solução é então, fazendo $a_2 = 0$,

$$u(t,r,\theta) = u_0(r,\theta) + \sum_{jn} r^{-1/2} J_{n+1/2}(\lambda_{nj} r) P_n(\cos\theta) \cos\, \lambda_{nj} v t \times$$
$$\times \frac{2}{a^2[J'_{n+1/2}(\lambda_{nj} a)]^2} \int_0^a dr\, r^{3/2} J_{n+1/2}(\lambda_{nj} r) \times$$
$$\times \frac{2n+1}{2} \int_0^\pi P_n(\cos\theta) sen\,\theta\, d\theta [g(r,\theta) - u_0(r,\theta)]\,.$$

4. Considere a região $a \leq r \leq b$. Calcule $u(t,r,\theta)$ sendo $u(t,a,\theta) = f(\theta)$, $u(t,b,\theta) = g(\theta)$, $u(0,r,\theta) = h(r,\theta)$.

(a) Temos agora,

$$\begin{aligned} u(t,r,\theta) = {} & u_0(r,\theta) + \sum_{\lambda n} r^{-1/2}[c_1 J_{n+1/2}(\lambda r) + c_2 Y_{n+1/2}(\lambda r)] \times \\ & \times P_n(\cos\theta)[a_1 \cos\, \lambda vt + a_2 sen\, \lambda vt] \, . \end{aligned}$$

As condições de contorno em $r = a$ e $r = b$ são,

$$\begin{aligned} u(t,a,\theta) = {} & f(\theta) = u_0(a,\theta) + \sum_{\lambda n} a^{-1/2}[c_1 J_{n+1/2}(\lambda a) + c_2 Y_{n+1/2}(\lambda a)] \times \\ & \times P_n(\cos\theta)[a_1 \cos\, \lambda vt + a_2 sen\, \lambda vt] \, , \\ u(t,b,\theta) = {} & g(\theta) = u_0(b,\theta) + \sum_{\lambda n} b^{-1/2}[c_1 J_{n+1/2}(\lambda b) + c_2 Y_{n+1/2}(\lambda b)] \times \\ & \times P_n(\cos\theta)[a_1 \cos\, \lambda vt + a_2 sen\, \lambda vt] \, . \end{aligned}$$

Escolhemos assim,

$$\begin{aligned} & c_1 J_{n+1/2}(\lambda a) + c_2 Y_{n+1/2}(\lambda a) = 0 \, , \\ & c_1 J_{n+1/2}(\lambda b) + c_2 Y_{n+1/2}(\lambda b) = 0 \, . \end{aligned}$$

Portanto, os valores de λ são definidos por,

$$J_{n+1/2}(\lambda_{nj} a) Y_{n+1/2}(\lambda_{nj} b) - Y_{n+1/2}(\lambda_{nj} a) J_{n+1/2}(\lambda_{nj} b) = 0 \, ,$$

ou,

$$u_{n+1/2}(\lambda_{nj} b) = 0 \, , \quad j = 1, 2, \ldots$$

com,

$$u_{n+1/2}(\lambda_{nj} r) \equiv Y_{n+1/2}(\lambda_{nj} a) J_{n+1/2}(\lambda_{nj} r) - J_{n+1/2}(\lambda_{nj} a) Y_{n+1/2}(\lambda_{nj} r) \, .$$

Temos então,

$$\begin{aligned} & f(\theta) = u_0(a,\theta) \, , \\ & g(\theta) = u_0(b,\theta) \, , \end{aligned}$$

e,

$$c_2 = -c_1 \frac{J_{n+1/2}(\lambda_{nj} a)}{Y_{n+1/2}(\lambda_{nj} a)} \, .$$

A solução fica assim, fazendo $c_1 = 1$,

$$u(t,r,\theta) = u_0(r,\theta) + \sum_{jn} r^{-1/2} u_{n+1/2}(\lambda_{nj} r) \frac{1}{Y_{n+1/2}(\lambda_{nj} a)} \times$$
$$\times P_n(\cos\theta)[a_1 \cos\, \lambda_{nj} vt + a_2 sen\, \lambda_{nj} vt]\,.$$

Escrevendo agora a condição inicial temos,

$$u(0,r,\theta) = h(r,\theta) = u_0(r,\theta) + \sum_{jn} r^{-1/2} u_{n+1/2}(\lambda_{nj} r) \frac{1}{Y_{n+1/2}(\lambda_{nj} a)} \times$$
$$\times P_n(\cos\theta) a_1\,,$$

ou,

$$h(r,\theta) - u_0(r,\theta) = \sum_{jn} r^{-1/2} u_{n+1/2}(\lambda_{nj} r) \frac{1}{Y_{n+1/2}(\lambda_{nj} a)} \times$$
$$\times P_n(\cos\theta) a_1\,.$$

A expressão acima é uma série de polinômios de Legendre, logo,

$$\sum_j r^{-1/2} u_{n+1/2}(\lambda_{nj} r) \frac{1}{Y_{n+1/2}(\lambda_{nj} a)} a_1 =$$
$$= \frac{2n+1}{2} \int_0^\pi [h(r,\theta) - u_0(r,\theta)] P_n(\cos\theta) sen\,\theta\, d\theta\,.$$

Temos agora uma série de funções de Bessel. Multiplicando por $r^{3/2} u_{n+1/2}(\lambda_{nk} r)$ *e integrando em* r,

$$\sum_j \int_a^b dr\, r u_{n+1/2}(\lambda_{nj} r) u_{n+1/2}(\lambda_{nk} r) \frac{1}{Y_{n+1/2}(\lambda_{nj} a)} a_1 =$$
$$= \int_a^b dr\, r^{3/2} u_{n+1/2}(\lambda_{nk} r) \times$$
$$\times \frac{2n+1}{2} \int_0^\pi [h(r,\theta) - u_0(r,\theta)] P_n(\cos\theta) sen\,\theta\, d\theta\,.$$

Usando a condição de ortogonalidade,

$$\int_a^b r u_{n+1/2}(\lambda_{nj} r) u_{n+1/2}(\lambda_{nk} r) dr = 0\,, \quad j \neq k\,,$$

temos,

$$
\begin{aligned}
&\int_a^b dr\, r u^2_{n+1/2}(\lambda_{nk} r)\frac{1}{Y_{n+1/2}(\lambda_{nk} a)} a_1 = \\
&= \int_a^b dr\, r^{3/2} u_{n+1/2}(\lambda_{nk} r) \times \\
&\times \frac{2n+1}{2}\int_0^\pi [h(r,\theta) - u_0(r,\theta)] P_n(\cos\theta) sen\,\theta\, d\theta\,.
\end{aligned}
$$

Os coeficientes a_1 são então,

$$
\begin{aligned}
&a_1 = \frac{Y_{n+1/2}(\lambda_{nk} a)}{\displaystyle\int_a^b dr\, r u^2_{n+1/2}(\lambda_{nk} r)} \times \\
&\times \int_a^b dr\, r^{3/2} u_{n+1/2}(\lambda_{nk} r) \times \\
&\times \frac{2n+1}{2}\int_0^\pi [h(r,\theta) - u_0(r,\theta)] P_n(\cos\theta) sen\,\theta\, d\theta\,.
\end{aligned}
$$

A solução é assim, fazendo $a_2 = 0$,

$$
\begin{aligned}
u(t,r,\theta) = u_0(r,\theta) + \sum_{jn} & r^{-1/2} u_{n+1/2}(\lambda_{nj} r) \times \\
&\times P_n(\cos\theta) \cos\, \lambda_{nj} v t \times \\
&\times \frac{1}{\displaystyle\int_a^b dr\, r u^2_{n+1/2}(\lambda_{nj} r)} \times \\
&\times \int_a^b dr\, r^{3/2} u_{n+1/2}(\lambda_{nj} r) \times \\
&\times \frac{2n+1}{2}\int_0^\pi [h(r,\theta) - u_0(r,\theta)] P_n(\cos\theta) sen\,\theta\, d\theta\,.
\end{aligned}
$$

5. Considere uma esfera de raio a. Calcule $u(t,r,\theta,\varphi)$ sendo $u(t,a,\theta,\varphi) = f(\theta,\varphi)$ e $u(0,r,\theta,\varphi) = g(r,\theta,\varphi)$.

(a) A solução finita é,

$$
\begin{aligned}
u(t,r,\theta,\varphi) = u_0(r,\theta,\varphi) + \sum_{\lambda n m} & r^{-1/2} J_{n+1/2}(\lambda r) P_n^m(\cos\theta) \times \\
&\times [b_1 \cos m\varphi + b_2 sen\, m\varphi] \\
&\times [a_1 \cos\, \lambda v t + a_2 sen\, \lambda v t]\,.
\end{aligned}
$$

A condição de contorno em $r = a$ nos dá,

$$
\begin{aligned}
u(t,a,\theta,\varphi) = f(\theta,\varphi) = u_0(a,\theta,\varphi) + \sum_{\lambda nm} a^{-1/2} J_{n+1/2}(\lambda a) P_n^m(\cos\theta) \times \\
\times[b_1 \cos m\varphi + b_2 sen\, m\varphi] \\
\times[a_1 \cos\, \lambda vt + a_2 sen\, \lambda vt]\,.
\end{aligned}
$$

Satisfazemos a equação acima escolhendo,

$$
J_{n+1/2}(\lambda_{nj} a) = 0\,, \quad j = 1, 2, \ldots
$$

A equação acima determina os valores possíveis de λ. Portanto,

$$
f(\theta,\varphi) = u_0(a,\theta,\varphi)\,.
$$

A condição inicial é, fazendo $a_1 = 1$

$$
\begin{aligned}
u(0,r,\theta,\varphi) = g(r,\theta,\varphi) = u_0(r,\theta,\varphi) + \sum_{jnm} r^{-1/2} J_{n+1/2}(\lambda_{nj} r) P_n^m(\cos\theta) \times \\
\times[b_1 \cos m\varphi + b_2 sen\, m\varphi]\,,
\end{aligned}
$$

ou,

$$
\begin{aligned}
g(r,\theta,\varphi) - u_0(r,\theta,\varphi) = \sum_{jnm} r^{-1/2} J_{n+1/2}(\lambda_{nj} r) P_n^m(\cos\theta) \times \\
\times[b_1 \cos m\varphi + b_2 sen\, m\varphi]\,,
\end{aligned}
$$

A expressão acima é uma série de Fourier, logo,

$$
\begin{aligned}
&\sum_{jn} r^{-1/2} J_{n+1/2}(\lambda_{nj} r) P_n(\cos\theta) b_{10} = \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} [g(r,\theta,x) - u_0(r,\theta,x)]\, dx\,, \\
&\sum_{jn} r^{-1/2} J_{n+1/2}(\lambda_{nj} r) P_n^m(\cos\theta) b_{1m} = \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} [g(r,\theta,x) - u_0(r,\theta,x)] \cos mx\, dx\,, \\
&\sum_{jn} r^{-1/2} J_{n+1/2}(\lambda_{nj} r) P_n^m(\cos\theta) b_{2m} = \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} [g(r,\theta,x) - u_0(r,\theta,x)] sen\, mx\, dx\,, \\
&m = 1, 2, \ldots
\end{aligned}
$$

Temos agora séries de polinômios de Legendre e de funções associadas de Legendre do primeiro tipo, assim,

$$\sum_j r^{-1/2} J_{n+1/2}(\lambda_{nj} r) b_{10} =$$
$$= \frac{2n+1}{2} \int_0^\pi P_n(\cos\theta) sen\,\theta\, d\theta \times$$
$$\times \frac{1}{2\pi} \int_{-\pi}^{\pi} [g(r,\theta,x) - u_0(r,\theta,x)]\, dx\,,$$
$$\sum_j r^{-1/2} J_{n+1/2}(\lambda_{nj} r) b_{1m} =$$
$$= \frac{(2n+1)(n-m)!}{2(n+m)!} \int_0^\pi P_n^m(\cos\theta) sen\,\theta\, d\theta \times$$
$$\times \frac{1}{\pi} \int_{-\pi}^{\pi} [g(r,\theta,x) - u_0(r,\theta,x)] \cos mx\, dx\,,$$
$$\sum_j r^{-1/2} J_{n+1/2}(\lambda_{nj} r) b_{2m} =$$
$$= \frac{(2n+1)(n-m)!}{2(n+m)!} \int_0^\pi P_n^m(\cos\theta) sen\,\theta\, d\theta \times$$
$$\times \frac{1}{\pi} \int_{-\pi}^{\pi} [g(r,\theta,x) - u_0(r,\theta,x)] sen\, mx\, dx\,,$$
$$m = 1, 2, \ldots$$

Temos agora séries de funções de Bessel. Multiplicando por $r^{3/2} J_{n+1/2}(\lambda_{nk} r)$ e integrando em r,

$$\sum_j \int_0^a dr\, r J_{n+1/2}(\lambda_{nj} r) J_{n+1/2}(\lambda_{nk} r) b_{10} =$$
$$= \int_0^a dr\, r^{3/2} J_{n+1/2}(\lambda_{nk} r) \frac{2n+1}{2} \int_0^\pi P_n(\cos\theta) sen\,\theta\, d\theta \times$$
$$\times \frac{1}{2\pi} \int_{-\pi}^{\pi} [g(r,\theta,x) - u_0(r,\theta,x)]\, dx\,,$$
$$\sum_j \int_0^a dr\, r J_{n+1/2}(\lambda_{nj} r) J_{n+1/2}(\lambda_{nk} r) b_{1m} =$$
$$= \int_0^a dr\, r^{3/2} J_{n+1/2}(\lambda_{nk} r) \frac{(2n+1)(n-m)!}{2(n+m)!} \int_0^\pi P_n^m(\cos\theta) sen\,\theta\, d\theta \times$$
$$\times \frac{1}{\pi} \int_{-\pi}^{\pi} [g(r,\theta,x) - u_0(r,\theta,x)] \cos mx\, dx\,,$$
$$\sum_j \int_0^a dr\, r J_{n+1/2}(\lambda_{nj} r) J_{n+1/2}(\lambda_{nk} r) b_{2m} =$$
$$= \int_0^a dr\, r^{3/2} J_{n+1/2}(\lambda_{nk} r) \frac{(2n+1)(n-m)!}{2(n+m)!} \int_0^\pi P_n^m(\cos\theta) sen\,\theta\, d\theta \times$$
$$\times \frac{1}{\pi} \int_{-\pi}^{\pi} [g(r,\theta,x) - u_0(r,\theta,x)] sen\, mx\, dx\,,$$
$$m = 1, 2, \ldots$$

Usando a relação de ortogonalidade,

$$\int_0^a r J_{n+1/2}(\lambda_{nj} r) J_{n+1/2}(\lambda_{nk} r) dr = \begin{cases} 0\,, & j \neq k\,, \\ \dfrac{a^2}{2}[J'_{n+1/2}(\lambda_{nk} a)]^2\,, & j = k\,, \end{cases}$$

temos,

$$\begin{aligned}
&\frac{a^2}{2}[J'_{n+1/2}(\lambda_{nk} a)]^2 b_{10} = \\
&= \int_0^a dr\, r^{3/2} J_{n+1/2}(\lambda_{nk} r) \frac{2n+1}{2} \int_0^\pi P_n(\cos\theta) sen\,\theta\, d\theta \times \\
&\times \frac{1}{2\pi} \int_{-\pi}^{\pi} [g(r,\theta,x) - u_0(r,\theta,x)]\, dx\,, \\
&\frac{a^2}{2}[J'_{n+1/2}(\lambda_{nk} a)]^2 b_{1m} = \\
&= \int_0^a dr\, r^{3/2} J_{n+1/2}(\lambda_{nk} r) \frac{(2n+1)(n-m)!}{2(n+m)!} \int_0^\pi P_n^m(\cos\theta) sen\,\theta\, d\theta \times \\
&\times \frac{1}{\pi} \int_{-\pi}^{\pi} [g(r,\theta,x) - u_0(r,\theta,x)] \cos mx\, dx\,, \\
&\frac{a^2}{2}[J'_{n+1/2}(\lambda_{nk} a)]^2 b_{2m} = \\
&= \int_0^a dr\, r^{3/2} J_{n+1/2}(\lambda_{nk} r) \frac{(2n+1)(n-m)!}{2(n+m)!} \int_0^\pi P_n^m(\cos\theta) sen\,\theta\, d\theta \times \\
&\times \frac{1}{\pi} \int_{-\pi}^{\pi} [g(r,\theta,x) - u_0(r,\theta,x)] sen\, mx\, dx\,, \\
&m = 1, 2, \ldots
\end{aligned}$$

e os coeficientes $b's$ são,

$$b_{10} = \frac{2}{a^2[J'_{n+1/2}(\lambda_{nk}a)]^2} \times$$
$$\times \int_0^a dr\, r^{3/2} J_{n+1/2}(\lambda_{nk} r) \frac{2n+1}{2} \int_0^\pi P_n(\cos\theta) sen\,\theta\, d\theta \times$$
$$\times \frac{1}{2\pi} \int_{-\pi}^{\pi} [g(r,\theta,x) - u_0(r,\theta,x)]\, dx\,,$$
$$b_{1m} = \frac{2}{a^2[J'_{n+1/2}(\lambda_{nk}a)]^2} \times$$
$$\times \int_0^a dr\, r^{3/2} J_{n+1/2}(\lambda_{nk} r) \frac{(2n+1)(n-m)!}{2(n+m)!} \int_0^\pi P_n^m(\cos\theta) sen\,\theta\, d\theta \times$$
$$\times \frac{1}{\pi} \int_{-\pi}^{\pi} [g(r,\theta,x) - u_0(r,\theta,x)] \cos mx\, dx\,,$$
$$b_{2m} = \frac{2}{a^2[J'_{n+1/2}(\lambda_{nk}a)]^2} \times$$
$$\times \int_0^a dr\, r^{3/2} J_{n+1/2}(\lambda_{nk} r) \frac{(2n+1)(n-m)!}{2(n+m)!} \int_0^\pi P_n^m(\cos\theta) sen\,\theta\, d\theta \times$$
$$\times \frac{1}{\pi} \int_{-\pi}^{\pi} [g(r,\theta,x) - u_0(r,\theta,x)] sen\, mx\, dx\,,$$
$$m = 1, 2, \ldots$$

A solução é então, fazendo $a_2 = 0$,

$$
\begin{aligned}
u(t,r,\theta,\varphi) = {} & u_0(r,\theta,\varphi) \\
& + \sum_{jn} r^{-1/2} J_{n+1/2}(\lambda_{nj} r) P_n(\cos\theta) \cos\, \lambda_{nj} vt \, \times \\
& \times \frac{2}{a^2 [J'_{n+1/2}(\lambda_{nk} a)]^2} \, \times \\
& \times \int_0^a dr\, r^{3/2} J_{n+1/2}(\lambda_{nk} r) \, \times \\
& \times \frac{2n+1}{2} \int_0^\pi P_n(\cos\theta) sen\,\theta \, d\theta \, \times \\
& \times \frac{1}{2\pi} \int_{-\pi}^{\pi} [g(r,\theta,x) - u_0(r,\theta,x)] \, dx \\
& + \sum_{jnm} r^{-1/2} J_{n+1/2}(\lambda_{nj} r) P_n^m(\cos\theta) \cos m\varphi \cos\, \lambda_{nj} vt \, \times \\
& \times \frac{2}{a^2 [J'_{n+1/2}(\lambda_{nk} a)]^2} \, \times \\
& \times \int_0^a dr\, r^{3/2} J_{n+1/2}(\lambda_{nk} r) \, \times \\
& \times \frac{(2n+1)(n-m)!}{2(n+m)!} \int_0^\pi P_n^m(\cos\theta) sen\,\theta \, d\theta \, \times \\
& \times \frac{1}{\pi} \int_{-\pi}^{\pi} [g(r,\theta,x) - u_0(r,\theta,x)] \cos mx \, dx \\
& + \sum_{jnm} r^{-1/2} J_{n+1/2}(\lambda_{nj} r) P_n^m(\cos\theta) sen\, m\varphi \cos\, \lambda_{nj} vt \, \times \\
& \times \frac{2}{a^2 [J'_{n+1/2}(\lambda_{nk} a)]^2} \, \times \\
& \times \int_0^a dr\, r^{3/2} J_{n+1/2}(\lambda_{nk} r) \, \times \\
& \times \frac{(2n+1)(n-m)!}{2(n+m)!} \int_0^\pi P_n^m(\cos\theta) sen\,\theta \, d\theta \, \times \\
& \times \frac{1}{\pi} \int_{-\pi}^{\pi} [g(r,\theta,x) - u_0(r,\theta,x)] sen\, mx \, dx \, .
\end{aligned}
$$

6. Considere a região $a \leq r \leq b$. Calcule $u(t,r,\theta,\varphi)$ sendo $u(t,a,\theta,\varphi) = f(\theta,\varphi)$, $u(t,b,\theta,\varphi) = g(\theta,\varphi)$, $u(0,r,\theta,\varphi) = h(r,\theta,\varphi)$.

(a) Escrevemos agora a solução como,

$$
\begin{aligned}
u(t,r,\theta,\varphi) = {} & u_0(r,\theta,\varphi) + \sum_{\lambda nm} r^{-1/2} [c_1 J_{n+1/2}(\lambda r) + c_2 Y_{n+1/2}(\lambda r)] \, \times \\
& \times P_n^m(\cos\theta) [b_1 \cos m\varphi + b_2 sen\, m\varphi] \\
& \times [a_1 \cos\, \lambda vt + a_2 sen\, \lambda vt] \, .
\end{aligned}
$$

As condições de contorno nos dão,

$$\begin{aligned}
u(t,a,\theta,\varphi) = f(\theta,\varphi) &= u_0(a,\theta,\varphi) \\
&+ \sum_{\lambda nm} a^{-1/2}[c_1 J_{n+1/2}(\lambda a) + c_2 Y_{n+1/2}(\lambda a)] \times \\
&\times P_n^m(\cos\theta)[b_1 \cos m\varphi + b_2 sen\, m\varphi] \\
&\times [a_1 \cos\,\lambda vt + a_2 sen\,\lambda vt]\,, \\
u(t,b,\theta,\varphi) = g(\theta,\varphi) &= u_0(b,\theta,\varphi) \\
&+ \sum_{\lambda nm} b^{-1/2}[c_1 J_{n+1/2}(\lambda b) + c_2 Y_{n+1/2}(\lambda b)] \times \\
&\times P_n^m(\cos\theta)[b_1 \cos m\varphi + b_2 sen\, m\varphi] \\
&\times [a_1 \cos\,\lambda vt + a_2 sen\,\lambda vt]\,.
\end{aligned}$$

Escolhemos assim,

$$\begin{aligned}
c_1 J_{n+1/2}(\lambda a) + c_2 Y_{n+1/2}(\lambda a) = 0\,, \\
c_1 J_{n+1/2}(\lambda b) + c_2 Y_{n+1/2}(\lambda b) = 0\,.
\end{aligned}$$

Portanto os valores de λ são determinados por,

$$J_{n+1/2}(\lambda a) Y_{n+1/2}(\lambda b) - Y_{n+1/2}(\lambda a) J_{n+1/2}(\lambda b) = 0\,,$$

ou,

$$u_{n+1/2}(\lambda_{nj} b) = 0\,, \quad j = 1, 2, \ldots$$

com,

$$u_{n+1/2}(\lambda_{nj} r) \equiv Y_{n+1/2}(\lambda_{nj} a) J_{n+1/2}(\lambda_{nj} r) - J_{n+1/2}(\lambda_{nj} a) Y_{n+1/2}(\lambda_{nj} r)\,.$$

Também temos,

$$\begin{aligned}
f(\theta,\varphi) = u_0(a,\theta,\varphi)\,, \\
g(\theta,\varphi) = u_0(b,\theta,\varphi)\,,
\end{aligned}$$

e,

$$c_2 = -c_1 \frac{J_{n+1/2}(\lambda a)}{Y_{n+1/2}(\lambda a)}\,.$$

A solução fica então, fazendo $c_1 = 1$,

$$
\begin{aligned}
u(t,r,\theta,\varphi) = u_0(r,\theta,\varphi) + \sum_{jnm} r^{-1/2} u_{n+1/2}(\lambda_{nj} r) \frac{1}{Y_{n+1/2}(\lambda_{nj} a)} \times \\
\times P_n^m(\cos\theta)[b_1 \cos m\varphi + b_2 sen\, m\varphi] \\
\times [a_1 \cos \lambda_{nj} vt + a_2 sen\, \lambda_{nj} vt] \,.
\end{aligned}
$$

A condição inicial é,

$$
\begin{aligned}
u(0,r,\theta,\varphi) = h(r,\theta,\varphi) = u_0(r,\theta,\varphi) \\
+ \sum_{jnm} r^{-1/2} u_{n+1/2}(\lambda_{nj} r) \frac{1}{Y_{n+1/2}(\lambda_{nj} a)} \times \\
\times P_n^m(\cos\theta)[b_1 \cos m\varphi + b_2 sen\, m\varphi] a_1 \,,
\end{aligned}
$$

ou, fazendo $a_1 = 1$,

$$
\begin{aligned}
h(r,\theta,\varphi) - u_0(r,\theta,\varphi) = \sum_{jnm} r^{-1/2} u_{n+1/2}(\lambda_{nj} r) \frac{1}{Y_{n+1/2}(\lambda_{nj} a)} \times \\
\times P_n^m(\cos\theta)[b_1 \cos m\varphi + b_2 sen\, m\varphi] \,.
\end{aligned}
$$

Temos uma série de Fourier, logo,

$$
\begin{aligned}
& \sum_{jn} r^{-1/2} u_{n+1/2}(\lambda_{nj} r) \frac{1}{Y_{n+1/2}(\lambda_{nj} a)} P_n(\cos\theta) b_{10} = \\
& = \frac{1}{2\pi} \int_{-\pi}^{\pi} [h(r,\theta,x) - u_0(r,\theta,x)] \, dx \,, \\
& \sum_{jn} r^{-1/2} u_{n+1/2}(\lambda_{nj} r) \frac{1}{Y_{n+1/2}(\lambda_{nj} a)} P_n^m(\cos\theta) b_{1m} = \\
& = \frac{1}{\pi} \int_{-\pi}^{\pi} [h(r,\theta,x) - u_0(r,\theta,x)] \cos mx \, dx \,, \\
& \sum_{jn} r^{-1/2} u_{n+1/2}(\lambda_{nj} r) \frac{1}{Y_{n+1/2}(\lambda_{nj} a)} P_n^m(\cos\theta) b_{2m} = \\
& = \frac{1}{\pi} \int_{-\pi}^{\pi} [h(r,\theta,x) - u_0(r,\theta,x)] sen\, mx \, dx \,.
\end{aligned}
$$

As expressões acima são séries de polinômios de Legendre e de funções associadas de Legendre, assim,

$$
\begin{aligned}
&\sum_j r^{-1/2} u_{n+1/2}(\lambda_{nj} r) \frac{1}{Y_{n+1/2}(\lambda_{nj} a)} b_{10} = \\
&= \frac{2n+1}{2} \int_0^{\pi} P_n(\cos\theta) sen\,\theta \, d\theta \times \\
&\times \frac{1}{2\pi} \int_{-\pi}^{\pi} [h(r,\theta,x) - u_0(r,\theta,x)] \, dx \,, \\
&\sum_j r^{-1/2} u_{n+1/2}(\lambda_{nj} r) \frac{1}{Y_{n+1/2}(\lambda_{nj} a)} b_{1m} = \\
&= \frac{(2n+1)(n-m)!}{2(n+m)!} \int_0^{\pi} P_n^m(\cos\theta) sen\,\theta \, d\theta \times \\
&\times \frac{1}{\pi} \int_{-\pi}^{\pi} [h(r,\theta,x) - u_0(r,\theta,x)] \cos mx \, dx \,, \\
&\sum_j r^{-1/2} u_{n+1/2}(\lambda_{nj} r) \frac{1}{Y_{n+1/2}(\lambda_{nj} a)} b_{2m} = \\
&= \frac{(2n+1)(n-m)!}{2(n+m)!} \int_0^{\pi} P_n^m(\cos\theta) sen\,\theta \, d\theta \times \\
&\times \frac{1}{\pi} \int_{-\pi}^{\pi} [h(r,\theta,x) - u_0(r,\theta,x)] sen\, mx \, dx \,.
\end{aligned}
$$

Temos agora séries de funções de Bessel. Multiplicando por $r^{3/2} u_{n+1/2}(\lambda_{nk} r)$ *e integrando em* r,

$$
\begin{aligned}
&\sum_j \int_a^b dr\, r u_{n+1/2}(\lambda_{nj} r) u_{n+1/2}(\lambda_{nk} r) \frac{1}{Y_{n+1/2}(\lambda_{nj} a)} b_{10} = \\
&= \int_a^b dr\, r^{3/2} u_{n+1/2}(\lambda_{nk} r) \frac{2n+1}{2} \int_0^{\pi} P_n(\cos\theta) sen\,\theta \, d\theta \times \\
&\times \frac{1}{2\pi} \int_{-\pi}^{\pi} [h(r,\theta,x) - u_0(r,\theta,x)] \, dx \,, \\
&\sum_j \int_a^b dr\, r u_{n+1/2}(\lambda_{nj} r) u_{n+1/2}(\lambda_{nk} r) \frac{1}{Y_{n+1/2}(\lambda_{nj} a)} b_{1m} = \\
&= \int_a^b dr\, r^{3/2} u_{n+1/2}(\lambda_{nk} r) \frac{(2n+1)(n-m)!}{2(n+m)!} \int_0^{\pi} P_n^m(\cos\theta) sen\,\theta \, d\theta \times \\
&\times \frac{1}{\pi} \int_{-\pi}^{\pi} [h(r,\theta,x) - u_0(r,\theta,x)] \cos mx \, dx \,, \\
&\sum_j \int_a^b dr\, r u_{n+1/2}(\lambda_{nj} r) u_{n+1/2}(\lambda_{nk} r) \frac{1}{Y_{n+1/2}(\lambda_{nj} a)} b_{2m} = \\
&= \int_a^b dr\, r^{3/2} u_{n+1/2}(\lambda_{nk} r) \frac{(2n+1)(n-m)!}{2(n+m)!} \int_0^{\pi} P_n^m(\cos\theta) sen\,\theta \, d\theta \times \\
&\times \frac{1}{\pi} \int_{-\pi}^{\pi} [h(r,\theta,x) - u_0(r,\theta,x)] sen\, mx \, dx \,.
\end{aligned}
$$

Usando a relação de ortogonalidade,

$$\int_a^b r u_{n+1/2}(\lambda_{nj} r) u_{n+1/2}(\lambda_{nk} r) dr = 0\,, \quad j \neq k\,,$$

obtemos,

$$\begin{aligned}
&\int_a^b dr\, r u_{n+1/2}^2(\lambda_{nk} r) \frac{1}{Y_{n+1/2}(\lambda_{nj} a)} b_{10} = \\
&= \int_a^b dr\, r^{3/2} u_{n+1/2}(\lambda_{nk} r) \frac{2n+1}{2} \int_0^\pi P_n(\cos\theta) sen\,\theta\, d\theta \times \\
&\times \frac{1}{2\pi} \int_{-\pi}^{\pi} [h(r,\theta,x) - u_0(r,\theta,x)]\, dx\,, \\
&\int_a^b dr\, r u_{n+1/2}^2(\lambda_{nk} r) \frac{1}{Y_{n+1/2}(\lambda_{nj} a)} b_{1m} = \\
&= \int_a^b dr\, r^{3/2} u_{n+1/2}(\lambda_{nk} r) \frac{(2n+1)(n-m)!}{2(n+m)!} \int_0^\pi P_n^m(\cos\theta) sen\,\theta\, d\theta \times \\
&\times \frac{1}{\pi} \int_{-\pi}^{\pi} [h(r,\theta,x) - u_0(r,\theta,x)] \cos mx\, dx\,, \\
&\int_a^b dr\, r u_{n+1/2}^2(\lambda_{nk} r) \frac{1}{Y_{n+1/2}(\lambda_{nj} a)} b_{2m} = \\
&= \int_a^b dr\, r^{3/2} u_{n+1/2}(\lambda_{nk} r) \frac{(2n+1)(n-m)!}{2(n+m)!} \int_0^\pi P_n^m(\cos\theta) sen\,\theta\, d\theta \times \\
&\times \frac{1}{\pi} \int_{-\pi}^{\pi} [h(r,\theta,x) - u_0(r,\theta,x)] sen\, mx\, dx\,.
\end{aligned}$$

Os coeficientes b's são então,

$$
\begin{aligned}
b_{10} = & \frac{Y_{n+1/2}(\lambda_{nj}a)}{\int_a^b dr\, r u_{n+1/2}^2(\lambda_{nk}r)} \times \\
& \times \int_a^b dr\, r^{3/2} u_{n+1/2}(\lambda_{nk}r) \frac{2n+1}{2} \int_0^\pi P_n(\cos\theta) sen\,\theta\, d\theta \times \\
& \times \frac{1}{2\pi} \int_{-\pi}^{\pi} [h(r,\theta,x) - u_0(r,\theta,x)]\, dx \,, \\
b_{1m} = & \frac{Y_{n+1/2}(\lambda_{nj}a)}{\int_a^b dr\, r u_{n+1/2}^2(\lambda_{nk}r)} \times \\
& \times \int_a^b dr\, r^{3/2} u_{n+1/2}(\lambda_{nk}r) \frac{(2n+1)(n-m)!}{2(n+m)!} \int_0^\pi P_n^m(\cos\theta) sen\,\theta\, d\theta \times \\
& \times \frac{1}{\pi} \int_{-\pi}^{\pi} [h(r,\theta,x) - u_0(r,\theta,x)] \cos mx\, dx \,, \\
b_{2m} = & \frac{Y_{n+1/2}(\lambda_{nj}a)}{\int_a^b dr\, r u_{n+1/2}^2(\lambda_{nk}r)} \times \\
& \times \int_a^b dr\, r^{3/2} u_{n+1/2}(\lambda_{nk}r) \frac{(2n+1)(n-m)!}{2(n+m)!} \int_0^\pi P_n^m(\cos\theta) sen\,\theta\, d\theta \times \\
& \times \frac{1}{\pi} \int_{-\pi}^{\pi} [h(r,\theta,x) - u_0(r,\theta,x)] sen\, mx\, dx \,.
\end{aligned}
$$

A solução é assim, fazendo $a_2 = 0$*,*

$$
\begin{aligned}
u(t,r,\theta,\varphi) = & \, u_0(r,\theta,\varphi) \\
& + \sum_{jnm} r^{-1/2} u_{n+1/2}(\lambda_{nj} r) P_n^m(\cos\theta) \cos\, \lambda_{nj} v t \,\times \\
& \times \frac{1}{\int_a^b dr\, r u_{n+1/2}^2(\lambda_{nj} r)} \,\times \\
& \times \int_a^b dr\, r^{3/2} u_{n+1/2}(\lambda_{nj} r) \frac{2n+1}{2} \int_0^\pi P_n(\cos\theta) sen\,\theta \, d\theta \,\times \\
& \times \frac{1}{2\pi} \int_{-\pi}^{\pi} [h(r,\theta,x) - u_0(r,\theta,x)] \, dx \\
& + \sum_{jnm} r^{-1/2} u_{n+1/2}(\lambda_{nj} r) P_n^m(\cos\theta) \cos\, \lambda_{nj} v t \cos m\varphi \,\times \\
& \times \frac{1}{\int_a^b dr\, r u_{n+1/2}^2(\lambda_{nj} r)} \,\times \\
& \times \int_a^b dr\, r^{3/2} u_{n+1/2}(\lambda_{nj} r) \frac{(2n+1)(n-m)!}{2(n+m)!} \int_0^\pi P_n^m(\cos\theta) sen\,\theta \, d\theta \,\times \\
& \times \frac{1}{\pi} \int_{-\pi}^{\pi} [h(r,\theta,x) - u_0(r,\theta,x)] \cos mx \, dx \\
& + \sum_{jnm} r^{-1/2} u_{n+1/2}(\lambda_{nj} r) P_n^m(\cos\theta) \cos\, \lambda_{nj} v t sen\, m\varphi \,\times \\
& \times \frac{1}{\int_a^b dr\, r u_{n+1/2}^2(\lambda_{nj} r)} \,\times \\
& \times \int_a^b dr\, r^{3/2} u_{n+1/2}(\lambda_{nj} r) \frac{(2n+1)(n-m)!}{2(n+m)!} \int_0^\pi P_n^m(\cos\theta) sen\,\theta \, d\theta \,\times \\
& \times \frac{1}{\pi} \int_{-\pi}^{\pi} [h(r,\theta,x) - u_0(r,\theta,x)] sen\, mx \, dx \,.
\end{aligned}
$$

7. Calcule $u(t,r,\theta,\varphi)$ em $0 \le r \le a$ com as condições,

$$
\begin{aligned}
& u(t,a,\theta,\varphi) = \mu(t,\theta,\varphi)\,, \\
& u(0,r,\theta,\varphi) = \varphi(r,\theta,\varphi)\,, \\
& u_t(0,r,\theta,\varphi) = \psi(r,\theta,\varphi)\,.
\end{aligned}
$$

8. Calcule $u(t,r,\theta,\varphi)$ em $a \le r \le b$ com as condições,

$$
\begin{aligned}
& u(t,a,\theta,\varphi) = \mu_1(t,\theta,\varphi)\,, \\
& u(t,b,\theta,\varphi) = \mu_2(t,\theta,\varphi)\,, \\
& u(0,r,\theta,\varphi) = \varphi(r,\theta,\varphi)\,, \\
& u_t(0,r,\theta,\varphi) = \psi(r,\theta,\varphi)\,.
\end{aligned}
$$

2 Apêndice

(a) Série de Fourier

$$
\begin{aligned}
f(x) &= \frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \mathrm{sen}\, mx)\,, \\
\frac{a_0}{2} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx\,, \\
a_m &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx\, dx\,, \\
b_m &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \mathrm{sen}\, mx\, dx\,, \quad m = 0, 1, 2, \ldots
\end{aligned}
$$

(b) Polinômios de Legendre ($x = \cos\theta$)

$$
\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= \frac{1}{2}(3x^2 - 1) \\
P_3(x) &= \frac{1}{2}(5x^3 - 3x) \\
P_4(x) &= \frac{1}{8}(35x^4 - 30x^2 + 3) \\
P_5(x) &= \frac{1}{8}(63x^5 - 70x^3 + 15x) \\
P_6(x) &= \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5) \\
P_7(x) &= \frac{1}{16}(497x^7 - 693x^5 + 315x^3 - 35x) \\
P_8(x) &= \frac{1}{128}(6435x^8 - 12012x^6 + 6930x^4 - 1260x^2 + 35)
\end{aligned}
$$

Relação de ortogonalidade:

$$
\int_{-1}^{+1} P_n(x) P_k(x) dx = \begin{cases} 0\,, & n \neq k\,, \\ \dfrac{2}{2n+1}\,, & n = k\,, \end{cases}
$$

(c) Funções associadas de Legendre

$$
\begin{aligned}
P_1^1(x) &= (1-x^2)^{1/2} = \operatorname{sen}\theta \\
P_2^1(x) &= 3x(1-x^2)^{1/2} = 3\operatorname{sen}\theta\cos\theta \\
P_2^2(x) &= 3(1-x^2) = 3\operatorname{sen}^2\theta \\
P_3^1(x) &= \frac{3}{2}(5x^2-1)(1-x^2)^{1/2} = \frac{3}{2}(5\cos^2\theta-1)\operatorname{sen}\theta \\
P_3^2(x) &= 15x(1-x^2) = 15\operatorname{sen}^2\theta\cos\theta \\
P_3^3(x) &= 15(1-x^2)^{3/2} = 15\operatorname{sen}^3\theta \\
P_4^1(x) &= \frac{5}{2}(7x^3-3x)(1-x^2)^{1/2} = \frac{5}{2}(7\cos^3\theta-3\cos\theta)\operatorname{sen}\theta \\
P_4^2(x) &= \frac{15}{2}(7x^2-1)(1-x^2) = \frac{15}{2}(7\cos^2\theta-1)\operatorname{sen}^2\theta \\
P_4^3(x) &= 105x(1-x^2)^{3/2} = 105\operatorname{sen}^3\theta\cos\theta \\
P_4^4(x) &= 105(1-x^2)^2 = 105\operatorname{sen}^4\theta
\end{aligned}
$$

Relação de ortogonalidade:

$$
\int_{-1}^{+1} P_n^m(x)P_k^m(x)dx = \begin{cases} 0\,, & n \neq k\,, \\ \dfrac{2}{2n+1}\dfrac{(n+m)!}{(n-m)!}\,, & n = k\,, \end{cases}
$$

(d) Série de polinômios de Legendre

$$
\begin{aligned}
f(\theta) &= \sum_{k=0}^{\infty} C_k P_k(\cos\theta)\,, \\
C_k &= \frac{2k+1}{2}\int_0^{\pi} f(\theta)P_k(\cos\theta)\operatorname{sen}\theta\, d\theta\,.
\end{aligned}
$$

(e) Série de funções associadas de Legendre do primeiro tipo

$$
\begin{aligned}
f(\theta) &= \sum_{k=0}^{\infty} D_k P_k^m(\cos\theta)\,, \\
D_k &= \frac{(2k+1)(k-m)!}{2(k+m)!}\int_0^{\pi} f(\theta)P_k^m(\cos\theta)\operatorname{sen}\theta\, d\theta\,,
\end{aligned}
$$

(f) Funções de Bessel

$$
\int_0^a rJ_{1/2}(\lambda_j r)J_{1/2}(\lambda_k r)dr = \begin{cases} 0\,, & j \neq k\,, \\ \dfrac{a^2}{2}[J'_{1/2}(\lambda_k a)]^2\,, & j = k\,, \end{cases}
$$

$$\int_0^a rJ_{n+1/2}(\lambda_{nj}r)J_{n+1/2}(\lambda_{nk}r)dr = \begin{cases} 0\,, & j \neq k\,, \\ \dfrac{a^2}{2}[J'_{n+1/2}(\lambda_{nk}a)]^2\,, & j = k\,, \end{cases}$$

$$u_{1/2}(\lambda_j r) \equiv Y_{1/2}(\lambda_j a)J_{1/2}(\lambda_j r) - J_{1/2}(\lambda_j a)Y_{1/2}(\lambda_j r)\,,$$

$$\int_a^b ru_{1/2}(\lambda_j r)u_{1/2}(\lambda_k r)dr = 0\,, \quad j \neq k\,.$$

Expansão em série de funções $u_{1/2}(\lambda_j r)$,

$$f(r) = \sum_j A_j u_{1/2}(\lambda_j r)\,,$$

$$A_j = \frac{\displaystyle\int_a^b rf(r)u_{1/2}(\lambda_j r)dr}{\displaystyle\int_a^b ru_{1/2}^2(\lambda_j r)dr}\,.$$

$$u_{n+1/2}(\lambda_{nj}r) \equiv Y_{n+1/2}(\lambda_{nj}a)J_{n+1/2}(\lambda_{nj}r) - J_{n+1/2}(\lambda_{nj}a)Y_{n+1/2}(\lambda_{nj}r)\,,$$

$$\int_a^b ru_{n+1/2}(\lambda_{nj}r)u_{n+1/2}(\lambda_{nk}r)dr = 0\,, \quad j \neq k\,.$$

Expansão em série de funções $u_{n+1/2}(\lambda_{nj}r)$,

$$f(r) = \sum_j A_j u_{n+1/2}(\lambda_{nj}r)\,,$$

$$A_j = \frac{\displaystyle\int_a^b rf(r)u_{n+1/2}(\lambda_{nj}r)dr}{\displaystyle\int_a^b ru_{n+1/2}^2(\lambda_{nj}r)dr}\,.$$

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