10 - A equação da onda em coordenadas cilíndricas

A equação da onda é,

$$\frac{\partial^2 u}{\partial t^2} = v^2 \nabla^2 u \,, \tag{1}$$

em que v é uma constante representando a velocidade de propagação da onda no meio considerado. Em coordenadas cilíndricas,

$$\frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial u}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2} \,. \tag{2}$$

Primeiro separamos o tempo das coordenadas espaciais, como fizemos na equação do calor, escrevendo a solução como,

$$u(t, \mathbf{r}) = T(t) F(\mathbf{r}) \,. \tag{3}$$

Substituindo em (1) obtemos,

$$\frac{1}{v^2} F \frac{d^2 T}{dt^2} = T \nabla^2 F \,,$$

ou,

$$\frac{1}{v^2 T}\frac{d^2 T}{dt^2} = \frac{\nabla^2 F}{F} \,.$$

Igualando a equação acima a uma constante,

$$\frac{1}{v^2 T}\frac{d^2 T}{dt^2} = \frac{\nabla^2 F}{F} = -\lambda^2 \,,$$

obtemos as equações,

$$\frac{d^2 T}{dt^2} + (\lambda v)^2 T = 0 \,, \tag{4}$$

e,

$$\nabla^2 F + \lambda^2 F = 0 \,. \tag{5}$$

A equação para T possui solução,

$$T(t) = a_1 \cos(\lambda v t) + a_2 \mathrm{sen}\,(\lambda v t) \,. \tag{6}$$

Notemos que a constante de separação λ está relacionada com as frequências de oscilação. De fato, temos,

$$\lambda v = \omega \,,$$

em que $\omega = 2\pi/T = 2\pi f$ é a frequência angular da onda, f a frequência linear, e T o período. Assim,

$$\lambda = \frac{\omega}{v} = \frac{\omega}{\lambda_c/T} = \frac{\omega T}{\lambda_c} = \frac{2\pi}{\lambda_c} \,,$$

em que λ_c é o comprimento da onda. Vemos que λ corresponde ao número de onda. Temos agora que resolver a equação para F. Como essa equação é idêntica à obtida no caso da equação do calor, não precisamos repetir o procedimento aqui. Podemos passar para os problemas.

1 Problemas

1. Considere a região $0 \leq \rho \leq a$. Calcule a solução $u(t, \rho)$ da equação de onda, sendo $u(t, a) = u_a$ e $u(0, \rho) = f(\rho)$.

A solução geral é,

$$u(t, \rho) = c_0 + d_0 \ln \rho + \sum_\lambda J_0(\lambda\rho)[a_1 \cos(\lambda v t) + a_2 sen\,(\lambda v t)] \,,$$

em que $c_0 + d_0 \ln \rho$ representa a solução da equação de Laplace. As condições de contorno nos dão as equações,

$$\begin{aligned} u(t, a) &= u_a = c_0 + d_0 u_0(a) + \sum_\lambda J_0(\lambda a)[a_1 \cos(\lambda v t) + a_2 sen\,(\lambda v t)] \,, \\ u(0, \rho) &= f(\rho) = c_0 + d_0 u_0(\rho) + \sum_\lambda J_0(\lambda\rho) a_1 \,. \end{aligned}$$

Satisfazemos as condições acima escolhendo,

$$J_0(\lambda_j a) = 0 \,, \quad j = 1, 2, \ldots$$

Obtemos assim as equações,

$$\begin{aligned} & u_a = c_0 + d_0 u_0(a) \,, \\ & f(\rho) - c_0 - d_0 u_0(\rho) = \sum_j J_0(\lambda_j \rho) a_{1j} \,. \end{aligned}$$

A segunda equação é uma série de funções de Bessel,

$$f(\rho) = \sum_{j=1} A_j J_n(\lambda_{jn}\rho)\,,$$
$$A_j = \frac{2}{a^2 J_{n+1}^2(\lambda_{jn}a)} \int_0^a \rho J_n(\lambda_{jn}\rho) f(\rho) d\rho\,,$$

logo,

$$a_{1j} = \frac{2}{a^2 J_1^2(\lambda_j a)} \int_0^a \rho J_0(\lambda_j \rho)[f(\rho) - c_0 - d_0 u_0(\rho)] d\rho\,,$$

em que usamos $\lambda_{j0} = \lambda_j$ por simplicidade. Como $f(a) = u_a$, estamos livres para escolher uma das constantes c_0 ou d_0. Escolhemos $c_0 = 0$ e $d_0 = 1$, assim,

$$u_a = u_0(a)\,,$$
$$f(\rho) - u_0(\rho) = \sum_j J_0(\lambda_j \rho) a_{1j}\,,$$

e,

$$a_{1j} = \frac{2}{a^2 J_1^2(\lambda_j a)} \int_0^a \rho J_0(\lambda_j \rho)[f(\rho) - u_0(\rho)] d\rho\,.$$

Também podemos fazer $a_2 = 0$. A solução é portanto,

$$u(t,\rho) = u_0(\rho) + \sum_j \cos(\lambda_j v t) \frac{2 J_0(\lambda_j \rho)}{a^2 J_1^2(\lambda_j a)} \times$$
$$\times \int_0^a \rho J_0(\lambda_j \rho)[f(\rho) - u_0(\rho)] d\rho\,.$$

Consideremos o caso particular em que $u_a = 0$. Nesse caso obtemos $u_0(a) = 0$ e portanto $u_0(\rho) = 0$. A solução fica então (Spiegel [7] probl. 6.94),

$$u(t,\rho) = \sum_j \cos(\lambda_j v t) \frac{2 J_0(\lambda_j \rho)}{a^2 J_1^2(\lambda_j a)} \int_0^a \rho J_0(\lambda_j \rho) f(\rho) d\rho\,.$$

2. Calcule $u(t,\rho)$ na região $0 \le \rho \le a$ com as condições,

$$
\begin{aligned}
u(t,a) &= \mu(t)\,,\\
u(0,\rho) &= \varphi(\rho)\,,\\
u_t(0,\rho) &= \psi(\rho)
\end{aligned}
$$

3. Considere a região $a \leq \rho \leq b$. Calcule $u(t,\rho)$ sendo $u(t,a) = u_a$, $u(t,b) = u_b$ e $u(0,\rho) = f(\rho)$ (Spiegel [7], probl. 6.117a com $u_a = u_b = 0$).

A solução é,

$$
\begin{aligned}
u(t,\rho) &= c_0 + d_0 u_0(\rho)\\
&\quad + \sum_\lambda [b_1 J_0(\lambda\rho) + b_2 Y_0(\lambda\rho)][a_1\cos(\lambda v t) + a_2 sen\,(\lambda v t)]\,,
\end{aligned}
$$

e as condições de contorno,

$$
\begin{aligned}
u(t,a) &= u_a = c_0 + d_0 u_0(a)\\
&\quad + \sum_\lambda [b_1 J_0(\lambda a) + b_2 Y_0(\lambda a)][a_1\cos(\lambda v t) + a_2 sen\,(\lambda v t)]\,,\\
u(t,b) &= u_b = c_0 + d_0 u_0(b)\\
&\quad + \sum_\lambda [b_1 J_0(\lambda b) + b_2 Y_0(\lambda b)][a_1\cos(\lambda v t) + a_2 sen\,(\lambda v t)]\,,\\
u(0,\rho) &= f(\rho) = c_0 + d_0 u_0(\rho)\\
&\quad + \sum_\lambda [b_1 J_0(\lambda\rho) + b_2 Y_0(\lambda\rho)]a_1\,.
\end{aligned}
$$

Satisfazemos as equações acima escolhendo,

$$
\begin{aligned}
b_{1j} J_0(\lambda_j a) + b_{2j} Y_0(\lambda_j a) = 0\,,\\
b_{1j} J_0(\lambda_j b) + b_{2j} Y_0(\lambda_j b) = 0\,,
\end{aligned}
$$

com λ_j satisfazendo,

$$
J_0(\lambda_j a)Y_0(\lambda_j b) - J_0(\lambda_j b)Y_0(\lambda_j a) = 0\,,\ \ j = 1, 2, \ldots
$$

Com isso podemos escrever,

$$
b_{2j} = -b_{1j}\frac{J_0(\lambda_j a)}{Y_0(\lambda_j a)}\,.
$$

Também fazemos $c_0 = a_2 = 0$, $d_0 = a_1 = 1$, *logo,*

$$
\begin{aligned}
&u_a = u_0(a)\\
&u_b = u_0(b)\\
&f(\rho) = u_0(\rho) + \sum_j u_0(\lambda_j\rho)\frac{b_{1j}}{Y_0(\lambda_j a)}\,,
\end{aligned}
$$

em que usamos,

$$u_0(\lambda_j\rho) \equiv J_0(\lambda_j\rho)Y_0(\lambda_j a) - J_0(\lambda_j a)Y_0(\lambda_j\rho)\,.$$

Uma série em funções de Bessel $u_0(\lambda_j\rho)$ *é,*

$$
\begin{aligned}
f(\rho) &= \sum_{j=1} A_j u_0(\lambda_j\rho)\,,\\
A_j &= \frac{\int_a^b \rho f(\rho)u_0(\lambda_j\rho)d\rho}{\int_a^b \rho[u_0(\lambda_j\rho)]^2 d\rho}\,,
\end{aligned}
$$

logo,

$$\frac{b_{1j}}{Y_0(\lambda_j a)} = \frac{1}{\int_a^b \rho[u_0(\lambda_j\rho)]^2 d\rho}\int_a^b \rho u_0(\lambda_j\rho)d\rho[f(\rho) - u_0(\rho)]\,,$$

ou,

$$b_{1j} = \frac{Y_0(\lambda_j a)}{\int_a^b \rho[u_0(\lambda_j\rho)]^2 d\rho}\int_a^b \rho u_0(\lambda_j\rho)d\rho[f(\rho) - u_0(\rho)]\,.$$

A solução é portanto,

$$
\begin{aligned}
u(t,\rho) &= u_0(\rho) + \sum_j u_0(\lambda_j\rho)\cos(\lambda_j vt)\,\times\\
&\times\frac{1}{\int_a^b \rho[u_0(\lambda_j\rho)]^2 d\rho}\int_a^b \rho u_0(\lambda_j\rho)d\rho[f(\rho) - u_0(\rho)]\,.
\end{aligned}
$$

4. Calcule $u(t,\rho)$ na região $a \le \rho \le b$ com as condições,

$$
\begin{aligned}
&u(t,a) = \mu_1(t)\,, \quad u(t,b) = \mu_2(t)\,,\\
&u(0,\rho) = \varphi(\rho)\,, \quad u_t(0,\rho) = \psi(\rho)\,.
\end{aligned}
$$

5. Considere a região $0 \leq \rho \leq a$ e $0 \leq z \leq L$. Calcule a solução $u(t,\rho,z)$ sendo $u(t,a,z) = f(z)$, $u(t,\rho,0) = g(\rho)$, $u(t,\rho,L) = h(\rho)$ e $u(0,\rho,z) = w(\rho,z)$.

Escrevemos a solução u como,

$$
\begin{aligned}
u(t,\rho,z) &= c_0 + d_0 u_0(\rho,z) \\
&\quad + \sum_{\lambda\mu} J_0(\alpha\rho)[b_1 \cos\mu z + b_2 sen\,\mu z][a_1 \cos(\lambda v t) + a_2 sen\,(\lambda v t)]\,, \\
&\quad \alpha = \sqrt{\lambda^2 - \mu^2}\,, \quad \lambda^2 > \mu^2\,,
\end{aligned}
$$

e as condições de contorno,

$$
\begin{aligned}
u(t,a,z) &= f(z) = c_0 + d_0 u_0(a,z) \\
&\quad + \sum_{\lambda\mu} J_0(\alpha a)[b_1 \cos\mu z + b_2 sen\,\mu z][a_1 \cos(\lambda v t) + a_2 sen\,(\lambda v t)]\,, \\
u(t,\rho,0) &= g(\rho) = c_0 + d_0 u_0(\rho,z) \\
&\quad + \sum_{\lambda\mu} J_0(\alpha\rho) b_1 [a_1 \cos(\lambda v t) + a_2 sen\,(\lambda v t)]\,, \\
u(t,\rho,L) &= h(\rho) = c_0 + d_0 u_0(\rho,z) \\
&\quad + \sum_{\lambda\mu} J_0(\alpha\rho)[b_1 \cos\mu L + b_2 sen\,\mu L][a_1 \cos(\lambda v t) + a_2 sen\,(\lambda v t)]\,, \\
u(0,\rho,z) &= w(\rho,z) = c_0 + d_0 u_0(\rho,z) \\
&\quad + \sum_{\lambda\mu} J_0(\alpha\rho)[b_1 \cos\mu z + b_2 sen\,\mu z] a_1\,, \\
&\quad \alpha = \sqrt{\lambda^2 - \mu^2}\,, \quad \lambda^2 > \mu^2\,.
\end{aligned}
$$

Satisfazemos as condições acima escolhendo,

$$
\begin{aligned}
& J_0(\alpha_j a) = 0\,, \quad j = 1,2,\ldots \\
& b_1 = 0\,, \quad b_2 = 1\,, \quad a_2 = 0\,, \\
& \mu_k L = k\pi\,, \quad k = 0,1,2,\ldots \\
& c_0 = 0\,, \quad d_0 = 1\,,
\end{aligned}
$$

logo,

$$
\begin{aligned}
f(z) &= u_0(a,z)\,, \\
g(\rho) &= u_0(\rho,0) \\
h(\rho) &= u_0(\rho,L) \\
w(\rho,z) &= u_0(\rho,z) + \sum_{jk} a_1 J_0(\alpha_j\rho) sen\, \mu_k z\,, \\
& \alpha = \sqrt{\lambda^2-\mu^2}\,,\quad \lambda^2 > \mu^2\,.
\end{aligned}
$$

A equação para w é uma série de Fourier de senos e uma série de funções de Bessel. Assim, como,

$$
\begin{aligned}
& f(z) = \sum_{j=1}^{\infty} B_j sen\,(j\pi z/L)\,, \\
& A_j = 0\,, \\
& B_j = \frac{2}{L}\int_0^L f(x) sen\,(j\pi x/L)\,dx\,,\quad j = 0,1,2,\ldots
\end{aligned}
$$

temos,

$$
\sum_j a_1 J_0(\alpha_j\rho) = \frac{2}{L}\int_0^L [w(\rho,z) - u_0(\rho,z)] sen\,(k\pi z/L)\,dz\,.
$$

A expressão acima é uma série de funções de Bessel,

$$
\begin{aligned}
f(\rho) &= \sum_{j=1} A_j J_n(\alpha_{jn}\rho)\,, \\
A_j &= \frac{2}{a^2 J_{n+1}^2(\alpha_{jn}a)}\int_0^a \rho J_n(\alpha_{jn}\rho) f(\rho) d\rho\,,
\end{aligned}
$$

assim, escrevendo $\alpha_{j0} = \alpha_j$ *por simplicidade,*

$$
a_1 = \frac{2}{a^2 J_1^2(\alpha_j a)}\int_0^a \rho J_0(\alpha_j\rho) d\rho \frac{2}{L}\int_0^L [w(\rho,z) - u_0(\rho,z)] sen\,(k\pi z/L)\,dz\,.
$$

A solução é então,

$$
\begin{aligned}
u(t,\rho,z) &= u_0(\rho,z) + \sum_{jk} J_0(\alpha_j\rho)\,sen\,\mu_k z \cos(\sqrt{\alpha_j^2+\mu_k^2}vt) \times \\
&\times \frac{2}{a^2 J_1^2(\alpha_j a)} \int_0^a \rho J_0(\alpha_j\rho) d\rho \times \\
&\times \frac{2}{L} \int_0^L [w(\rho,z) - u_0(\rho,z)]\,sen\,(k\pi z/L)\, dz\,, \\
&\alpha = \sqrt{\lambda^2-\mu^2}\,, \quad \lambda^2 > \mu^2\,.
\end{aligned}
$$

6. Considere a região $0 \le \rho \le a$ e $0 \le z \le L$. Calcule a solução $u(t,\rho,z)$ sendo,

$$
\begin{aligned}
&u(t,a,z) = \mu_1(t,z)\,, \quad u(t,\rho,0) = \nu_1(t,\rho)\,, \quad u(t,\rho,L) = \nu_2(t,\rho)\,, \\
&u(0,\rho,z) = \varphi(\rho,z)\,, \quad u_t(0,\rho,z) = \psi(\rho,z)\,.
\end{aligned}
$$

7. Considere a região $a \le \rho \le b$ e $0 \le z \le L$. Calcule a solução $u(t,\rho,z)$ sendo $u(t,a,z) = f(z)$, $u(t,b,z) = g(z)$, $u(t,\rho,0) = h(\rho)$, $u(t,\rho,L) = v(\rho)$ e $u(0,\rho,z) = w(\rho,z)$.

Escrevemos u como,

$$
\begin{aligned}
u(t,\rho,z) &= c_0 + d_0 u_0(\rho,z) \\
&+ \sum_{\lambda\mu} [c_1 J_0(\alpha\rho) + c_2 Y_0(\alpha\rho)] \times \\
&\times [b_1 \cos\mu z + b_2\,sen\,\mu z][a_1 \cos(\lambda vt) + a_2\,sen\,(\lambda vt)]\,, \\
&\alpha = \sqrt{\lambda^2-\mu^2}\,, \quad \lambda^2 > \mu^2\,,
\end{aligned}
$$

e as condições de contorno,

$$
\begin{aligned}
u(t,a,z) &= f(z) = c_0 + d_0 u_0(a,z) \\
&+ \sum_{\lambda\mu} [c_1 J_0(\alpha a) + c_2 Y_0(\alpha a)] \times \\
&\times [b_1 \cos \mu z + b_2 sen\, \mu z][a_1 \cos(\lambda v t) + a_2 sen\,(\lambda v t)]\,, \\
u(t,b,z) &= g(z) = c_0 + d_0 u_0(b,z) \\
&+ \sum_{\lambda\mu} [c_1 J_0(\alpha b) + c_2 Y_0(\alpha b)] \times \\
&\times [b_1 \cos \mu z + b_2 sen\, \mu z][a_1 \cos(\lambda v t) + a_2 sen\,(\lambda v t)]\,, \\
u(t,\rho,0) &= h(\rho) = c_0 + d_0 u_0(\rho,0) \\
&+ \sum_{\lambda\mu} [c_1 J_0(\alpha \rho) + c_2 Y_0(\alpha \rho)] \times \\
&\times b_1 [a_1 \cos(\lambda v t) + a_2 sen\,(\lambda v t)]\,, \\
u(t,\rho,L) &= v(\rho) = c_0 + d_0 u_0(\rho,L) \\
&+ \sum_{\lambda\mu} [c_1 J_0(\alpha \rho) + c_2 Y_0(\alpha \rho)] \times \\
&\times [b_1 \cos \mu L + b_2 sen\, \mu L][a_1 \cos(\lambda v t) + a_2 sen\,(\lambda v t)]\,, \\
u(0,\rho,z) &= w(\rho,z) = c_0 + d_0 u_0(\rho,z) \\
&+ \sum_{\lambda\mu} [c_1 J_0(\alpha \rho) + c_2 Y_0(\alpha \rho)] \times \\
&\times [b_1 \cos \mu z + b_2 sen\, \mu z] a_1\,, \\
&\alpha = \sqrt{\lambda^2 - \mu^2}\,, \quad \lambda^2 > \mu^2\,,
\end{aligned}
$$

Escolhemos então,

$$
\begin{aligned}
& c_1 J_0(\alpha a) + c_2 Y_0(\alpha a) = 0\,, \\
& c_1 J_0(\alpha b) + c_2 Y_0(\alpha b) = 0\,, \\
& b_1 = 0\,, \quad b_2 = 1\,, \\
& \mu_k L = k\pi\,, \\
& c_0 = 0\,, \quad d_0 = 1\,,
\end{aligned}
$$

assim, os valores de α *são determinados por,*

$$
J_0(\alpha_j a) Y_0(\alpha_j b) - J_0(\alpha_j b) Y_0(\alpha_j a) = 0\,, \quad j = 1, 2, \ldots
$$

e podemos escrever,

$$c_2 = -c_1 \frac{J_0(\alpha a)}{Y_0(\alpha a)}\,.$$

As condições de contorno ficam, fazendo $c_1 = 1$*,*

$$\begin{aligned}
f(z) &= u_0(a,z)\,,\\
g(z) &= u_0(b,z)\,,\\
h(\rho) &= u_0(\rho,0)\,,\\
v(\rho) &= u_0(\rho,L)\,,\\
w(\rho,z) &= u_0(\rho,z) + \sum_{jk} a_1 \frac{1}{Y_0(\alpha_j a)} u_0(\alpha_j\rho)\, sen\, \mu_k z\,,\\
&\alpha = \sqrt{\lambda^2-\mu^2}\,,\quad \lambda^2 > \mu^2\,,
\end{aligned}$$

com,

$$u_0(\alpha_j\rho) \equiv J_0(\alpha_j\rho)Y_0(\alpha_j a) - J_0(\alpha_j a)Y_0(\alpha_j\rho)\,.$$

Temos novamente séries de Fourier e de Bessel. Assim, uma série de Fourier de senos é,

$$\begin{aligned}
&f(z) = \sum_{j=1}^{\infty} B_j\, sen\,(j\pi z/L)\,,\\
&A_j = 0\,,\\
&B_j = \frac{2}{L}\int_0^L f(x)\, sen\,(j\pi x/L)\, dx\,,\quad j = 1,2,\ldots
\end{aligned}$$

logo,

$$\sum_j a_1 \frac{1}{Y_0(\alpha_j a)} u_0(\alpha_j\rho) = \frac{2}{L}\int_0^L sen\,(k\pi z/L)\, dz[w(\rho,z) - u_0(\rho,z)]\,.$$

Uma série de funções de Bessel $u_\mu(\alpha\rho)$ *é,*

$$\begin{aligned}
f(\rho) &= \sum_{j=1} A_j u_\mu(\alpha_{j\mu}\rho)\,,\\
A_j &= \frac{\int_a^b \rho f(\rho) u_\mu(\alpha_{j\mu}\rho)d\rho}{\int_a^b \rho[u_\mu(\alpha_{j\mu}\rho)]^2 d\rho}\,,
\end{aligned}$$

assim,

$$a_1 \frac{1}{Y_0(\alpha_j a)} = \frac{1}{\int_a^b \rho[u_0(\alpha_j \rho)]^2 d\rho} \int_a^b \rho u_0(\alpha_j \rho) d\rho \times$$
$$\times \frac{2}{L} \int_0^L sen\,(k\pi z/L)\, dz[w(\rho,z) - u_0(\rho,z)]\,,$$

ou,

$$a_1 = \frac{Y_0(\alpha_j a)}{\int_a^b \rho[u_0(\alpha_j \rho)]^2 d\rho} \int_a^b \rho u_0(\alpha_j \rho) d\rho \times$$
$$\times \frac{2}{L} \int_0^L sen\,(k\pi z/L)\, dz[w(\rho,z) - u_0(\rho,z)]\,,$$

A solução é então, escolhendo $a_2 = 0$*,*

$$\begin{aligned} u(t,\rho,z) &= u_0(\rho,z) \\ &+ \sum_{jk} u_0(\alpha_j \rho) sen\,\mu_k z \cos(\sqrt{\alpha_j^2 + \mu_k^2} vt) \times \\ &\times \frac{1}{\int_a^b \rho[u_0(\alpha_j \rho)]^2 d\rho} \int_a^b \rho u_0(\alpha_j \rho) d\rho \times \\ &\times \frac{2}{L} \int_0^L sen\,(k\pi z/L)\, dz[w(\rho,z) - u_0(\rho,z)]\,, \\ &\alpha = \sqrt{\lambda^2 - \mu^2}\,, \quad \lambda^2 > \mu^2\,. \end{aligned}$$

8. Considere a região $a \leq \rho \leq b$ e $0 \leq z \leq L$. Calcule a solução $u(t,\rho,z)$ sendo,

$$u(t,a,z) = \mu_1(t,z)\,, \quad u(t,b,z) = \mu_2(t,z)\,,$$
$$u(t,\rho,0) = \nu_1(t,\rho)\,, \quad u(t,\rho,L) = \nu_2(t,\rho)\,,$$
$$u(0,\rho,z) = \varphi(\rho,z)\,, \quad u_t(0,\rho,z) = \psi(\rho,z)\,.$$

9. Calcule $u(t,\rho,\varphi)$ em $0 \leq \rho \leq a$ com as condições de contorno $u(t,a,\varphi) = f(\varphi)$, $u(0,\rho,\varphi) = g(\rho,\varphi)$ (Spiegel [7], probl. 6.31 com $f = 0$).

A solução é,

$$
\begin{aligned}
u(t,\rho,\varphi) &= c_0 + d_0 u_0(\rho,\varphi) \\
&\quad + \sum_{\lambda\mu} J_\mu(\lambda\rho)[b_{1\mu}\cos\mu\varphi + b_{2\mu} sen\,\mu\varphi] \times \\
&\quad \times [a_1\cos(\lambda v t) + a_2 sen\,(\lambda v t)]\,,
\end{aligned}
$$

e as condições de contorno são,

$$
\begin{aligned}
u(t,a,\varphi) &= f(\varphi) = c_0 + d_0 u_0(a,\varphi) \\
&\quad + \sum_{\lambda\mu} J_\mu(\lambda a)[b_{1\mu}\cos\mu\varphi + b_{2\mu} sen\,\mu\varphi] \times \\
&\quad \times [a_1\cos(\lambda v t) + a_2 sen\,(\lambda v t)]\,, \\
u(0,\rho,\varphi) &= g(\rho,\varphi) = c_0 + d_0 u_0(\rho,\varphi) \\
&\quad + \sum_{\lambda\mu} J_\mu(\lambda\rho)[b_{1\mu}\cos\mu\varphi + b_{2\mu} sen\,\mu\varphi] a_1\,.
\end{aligned}
$$

Escolhemos portanto,

$$
\begin{aligned}
&c_0 = 0\,, \quad d_0 = 1\,, \\
&J_\mu(\lambda_{j\mu} a) = 0\,, \quad j = 1,2,\ldots \\
&a_1 = 1\,, a_2 = 0\,.
\end{aligned}
$$

Com isso as condições de contorno ficam,

$$
\begin{aligned}
f(\varphi) &= u_0(a,\varphi)\,, \\
g(\rho,\varphi) &= u_0(\rho,\varphi) + \sum_{j\mu} J_\mu(\lambda_{j\mu}\rho)[b_{1\mu}\cos\mu\varphi + b_{2\mu} sen\,\mu\varphi]\,.
\end{aligned}
$$

A última expressão acima é uma série de Fourier e uma série de Bessel. Temos,

$$
\begin{aligned}
&f(\varphi) = \frac{A_0}{2} + \sum_{j=1}^{\infty}(A_j\cos j\varphi + B_j sen\,j\varphi)\,, \\
&A_j = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos jx\,dx\,, \\
&B_j = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) sen\,jx\,dx\,, \quad j = 0,1,2,\ldots
\end{aligned}
$$

logo,

$$\begin{aligned}
\frac{A_0(\rho)}{2} &= \sum_j J_0(\lambda_{j0}\rho) b_{10} = \frac{1}{2\pi}\int_{-\pi}^{\pi} dx[g(\rho,x)-u_0(\rho,x)]\,,\\
A_\mu(\rho) &= \sum_j J_\mu(\lambda_{j\mu}\rho) b_{1\mu} = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos\mu x\, dx[g(\rho,x)-u_0(\rho,x)]\,,\\
B_\mu(\rho) &= \sum_j J_\mu(\lambda_{j\mu}\rho) b_{2\mu} = \frac{1}{\pi}\int_{-\pi}^{\pi} sen\,\mu x\, dx[g(\rho,x)-u_0(\rho,x)]\,.
\end{aligned}$$

As expressões acima são séries de funções de Bessel,

$$\begin{aligned}
f(\rho) &= \sum_{j=1} A_j J_n(\lambda_{jn}\rho)\,,\\
A_j &= \frac{2}{a^2 J_{n+1}^2(\lambda_{jn}a)}\int_0^a \rho J_n(\lambda_{jn}\rho) f(\rho) d\rho\,,
\end{aligned}$$

assim,

$$\begin{aligned}
b_{10} &= \frac{2}{a^2 J_1^2(\lambda_{j0}a)}\int_0^a \rho J_0(\lambda_{j0}\rho) d\rho \frac{A_0(\rho)}{2}\,,\\
b_{1\mu} &= \frac{2}{a^2 J_{\mu+1}^2(\lambda_{j\mu}a)}\int_0^a \rho J_\mu(\lambda_{j\mu}\rho) A_\mu(\rho) d\rho\,,\\
b_{2\mu} &= \frac{2}{a^2 J_{\mu+1}^2(\lambda_{j\mu}a)}\int_0^a \rho J_\mu(\lambda_{j\mu}\rho) B_\mu(\rho) d\rho\,,
\end{aligned}$$

ou,

$$\begin{aligned}
b_{10} &= \frac{2}{a^2 J_1^2(\lambda_{j0}a)}\int_0^a \rho J_0(\lambda_{j0}\rho) d\rho \frac{1}{2\pi}\int_{-\pi}^{\pi} dx[g(\rho,x)-u_0(\rho,x)]\,,\\
b_{1\mu} &= \frac{2}{a^2 J_{\mu+1}^2(\lambda_{j\mu}a)}\int_0^a \rho J_\mu(\lambda_{j\mu}\rho) d\rho \frac{1}{\pi}\int_{-\pi}^{\pi} \cos\mu x\, dx[g(\rho,x)-u_0(\rho,x)]\,,\\
b_{2\mu} &= \frac{2}{a^2 J_{\mu+1}^2(\lambda_{j\mu}a)}\int_0^a \rho J_\mu(\lambda_{j\mu}\rho) d\rho \frac{1}{\pi}\int_{-\pi}^{\pi} sen\,\mu x\, dx[g(\rho,x)-u_0(\rho,x)]\,.
\end{aligned}$$

A solução é portanto,

$$
\begin{aligned}
u(t,\rho,\varphi) &= u_0(\rho,\varphi) \\
&+ \sum_j J_0(\lambda_j\rho)\cos(\lambda_j vt)\times \\
&\times \frac{2}{a^2J_1^2(\lambda_{j0}a)}\int_0^a \rho J_0(\lambda_{j0}\rho)d\rho\times \\
&\times \frac{1}{2\pi}\int_{-\pi}^{\pi} dx[g(\rho,x)-u_0(\rho,x)] \\
&+ \sum_{j\mu} J_\mu(\lambda_{j\mu}\rho)\cos\mu\varphi\cos(\lambda_{j\mu}vt)\times \\
&\times \frac{2}{a^2J_{\mu+1}^2(\lambda_{j\mu}a)}\int_0^a \rho J_\mu(\lambda_{j\mu}\rho)d\rho\times \\
&\times \frac{1}{\pi}\int_{-\pi}^{\pi} \cos\mu x\, dx[g(\rho,x)-u_0(\rho,x)] \\
&+ \sum_{j\mu} J_\mu(\lambda_{j\mu}\rho) sen\,\mu\varphi\cos(\lambda_{j\mu}vt)\times \\
&\times \frac{2}{a^2J_{\mu+1}^2(\lambda_{j\mu}a)}\int_0^a \rho J_\mu(\lambda_{j\mu}\rho)d\rho\times \\
&\times \frac{1}{\pi}\int_{-\pi}^{\pi} sen\,\mu x\, dx[g(\rho,x)-u_0(\rho,x)]\,.
\end{aligned}
$$

Consideremos o caso particular $u(t,a,\varphi)=f(\varphi)=0$, $u(0,\rho,\varphi)=g(\rho,\varphi)=\rho\cos 3\varphi$, $u_t(0,\rho,\varphi)=0$ *(Spiegel [7], probl. 6.95). Temos então* $u_0(\rho,\varphi)=0$ *e,*

$$
\begin{aligned}
u(t,\rho,\varphi) &= \sum_j J_0(\lambda_j\rho)\cos(\lambda_j vt)\times \\
&\times\frac{2}{a^2J_1^2(\lambda_{j0}a)}\int_0^a \rho J_0(\lambda_{j0}\rho)d\rho\times \\
&\times\frac{1}{2\pi}\int_{-\pi}^{\pi} dx[\rho\cos 3x] \\
&+\sum_{j\mu} J_\mu(\lambda_{j\mu}\rho)\cos\mu\varphi\cos(\lambda_{j\mu}vt)\times \\
&\times\frac{2}{a^2J_{\mu+1}^2(\lambda_{j\mu}a)}\int_0^a \rho J_\mu(\lambda_{j\mu}\rho)d\rho\times \\
&\times\frac{1}{\pi}\int_{-\pi}^{\pi}\cos\mu x\,dx[\rho\cos 3x] \\
&+\sum_{j\mu} J_\mu(\lambda_{j\mu}\rho)sen\,\mu\varphi\cos(\lambda_{j\mu}vt)\times \\
&\times\frac{2}{a^2J_{\mu+1}^2(\lambda_{j\mu}a)}\int_0^a \rho J_\mu(\lambda_{j\mu}\rho)d\rho\times \\
&\times\frac{1}{\pi}\int_{-\pi}^{\pi} sen\,\mu x\,dx[\rho\cos 3x]\,.
\end{aligned}
$$

Usando,

$$
\begin{aligned}
&\int_{-\pi}^{\pi} dx\cos 3x = 0\,, \\
&\int_{-\pi}^{\pi}\cos\mu x\ \cos 3x\,dx = \begin{cases}\pi\,, & \mu=3\\ 0\,, & \mu\neq 3\end{cases} \\
&\int_{-\pi}^{\pi} sen\,\mu x\ \cos 3x\,dx = 0\,.
\end{aligned}
$$

obtemos,

$$
\begin{aligned}
u(t,\rho,\varphi) &= \sum_j J_3(\lambda_{j3}\rho)\cos 3\varphi\cos(\lambda_{j3}vt)\times \\
&\times\frac{2}{a^2J_4^2(\lambda_{j3}a)}\int_0^a \rho^2 J_3(\lambda_{j3}\rho)d\rho\,.
\end{aligned}
$$

Usando a integral indefinida,

$$\int \rho^2 J_3(\lambda\rho)d\rho \;\; = \;\; -\frac{\rho^2}{\lambda}J_2(\lambda\rho) - \frac{4\rho}{\lambda^2}J_1(\lambda\rho) - \frac{8}{\lambda^3}J_0(\lambda\rho)\,,$$

temos,

$$\int_0^a \rho^2 J_3(\lambda\rho)d\rho \;\; = \;\; \frac{1}{\lambda^3}\left[-\lambda^2 a^2 J_2(\lambda a) - 4\lambda a J_1(\lambda a) - 8J_0(\lambda a) + 8\right]\,.$$

Por outro lado, temos a relação,

$$J_2(x) = \frac{2}{x}J_1(x) - J_0(x)\,,$$

logo,

$$\int_0^a \rho^2 J_3(\lambda\rho)d\rho \;\; = \;\; \frac{1}{\lambda^3}\left[-6\lambda a J_1(\lambda a) + (\lambda^2 a^2 - 8)J_0(\lambda a) + 8\right]\,.$$

Podemos escrever então,

$$\begin{aligned} u(t,\rho,\varphi) \;\; &= \;\; \sum_j J_3(\lambda_{j3}\rho)\cos 3\varphi \cos(\lambda_{j3}vt)\,\times \\ &\times \frac{2}{a^2 J_4^2(\lambda_{j3}a)}\frac{1}{\lambda_{j3}^3}\left[-6\lambda_{j3}aJ_1(\lambda_{j3}a) + (\lambda_{j3}^2 a^2 - 8)J_0(\lambda_{j3}a) + 8\right]\,. \end{aligned}$$

10. Calcule $u(t,\rho,\varphi)$ em $0 \le \rho \le a$ com as condições de contorno,

$$\begin{aligned} u(t,a,\varphi) &= \mu(t,\varphi)\,, \\ u(0,\rho,\varphi) &= \varphi(\rho,\varphi)\,, \\ u_t(0,\rho,\varphi) &= \psi(\rho,\varphi)\,. \end{aligned}$$

11. Calcule $u(t,\rho,\varphi)$ em $a \le \rho \le b$ com as condições de contorno,

$$\begin{aligned} u(t,a,\varphi) &= \mu_1(t,\varphi)\,, \\ u(t,b,\varphi) &= \mu_2(t,\varphi)\,, \\ u(0,\rho,\varphi) &= \varphi(\rho,\varphi)\,, \\ u_t(0,\rho,\varphi) &= \psi(\rho,\varphi)\,. \end{aligned}$$

12. Calcule $u(t,\rho,\varphi)$ em $a \leq \rho \leq b$ com as condições de contorno $u(t,a,\varphi) = f(\varphi)$, $u(t,b,\varphi) = g(\varphi)$, $u(0,\rho,\varphi) = h(\rho,\varphi)$ (Spiegel [7], probl. 6.117b com $f = g = 0$).

A solução é,

$$\begin{aligned} u(t,\rho,\varphi) &= c_0 + d_0 u_0(\rho,\varphi) \\ &+ \sum_{\lambda\mu} [c_{1\lambda} J_\mu(\lambda\rho) + c_{2\lambda} Y_\mu(\lambda\rho)][b_{1\mu}\cos\mu\varphi + b_{2\mu} sen\,\mu\varphi] \times \\ &\times [a_1 \cos(\lambda v t) + a_2 sen\,(\lambda v t)]\,, \end{aligned}$$

e as condições de contorno,

$$\begin{aligned} u(t,a,\varphi) &= f(\varphi) = c_0 + d_0 u_0(a,\varphi) \\ &+ \sum_{\lambda\mu} [c_{1\lambda} J_\mu(\lambda a) + c_{2\lambda} Y_\mu(\lambda a)][b_{1\mu}\cos\mu\varphi + b_{2\mu} sen\,\mu\varphi] \times \\ &\times [a_1 \cos(\lambda v t) + a_2 sen\,(\lambda v t)]\,, \\ u(t,b,\varphi) &= g(\varphi) = c_0 + d_0 u_0(b,\varphi) \\ &+ \sum_{\lambda\mu} [c_{1\lambda} J_\mu(\lambda b) + c_{2\lambda} Y_\mu(\lambda b)][b_{1\mu}\cos\mu\varphi + b_{2\mu} sen\,\mu\varphi] \times \\ &\times [a_1 \cos(\lambda v t) + a_2 sen\,(\lambda v t)]\,, \\ u(0,\rho,\varphi) &= h(\rho,\varphi) = c_0 + d_0 u_0(\rho,\varphi) \\ &+ \sum_{\lambda\mu} [c_{1\lambda} J_\mu(\lambda\rho) + c_{2\lambda} Y_\mu(\lambda\rho)][b_{1\mu}\cos\mu\varphi + b_{2\mu} sen\,\mu\varphi] \times \\ &\times a_1\,. \end{aligned}$$

Escolhemos,

$$\begin{aligned} &c_{1\lambda} J_\mu(\lambda a) + c_{2\lambda} Y_\mu(\lambda a) = 0\,, \\ &c_{1\lambda} J_\mu(\lambda b) + c_{2\lambda} Y_\mu(\lambda b) = 0\,, \\ &c_0 = 0\,, \quad d_0 = 1\,, \\ &a_1 = 1\,, \quad a_2 = 0\,. \end{aligned}$$

Podemos então escrever c_2 como,

$$c_{2\lambda} = -c_{1\lambda} \frac{J_\mu(\lambda a)}{Y_\mu(\lambda a)}\,,$$

logo, fazendo $c_{1\lambda} = 1$,

$$
\begin{aligned}
f(\varphi) &= u_0(a,\varphi) \\
g(\varphi) &= u_0(b,\varphi) \\
h(\rho,\varphi) &= u_0(\rho,\varphi) \\
&\quad + \sum_{\lambda\mu} [J_\mu(\lambda\rho)Y_\mu(\lambda a) - J_\mu(\lambda a)Y_\mu(\lambda\rho)] \frac{1}{Y_\mu(\lambda a)} \times \\
&\quad \times [b_{1\mu} \cos\mu\varphi + b_{2\mu} sen\, \mu\varphi]\,.
\end{aligned}
$$

Usando,

$$
u_\mu(\lambda\rho) \equiv J_\mu(\lambda\rho)Y_\mu(\lambda a) - J_\mu(\lambda a)Y_\mu(\lambda\rho)\,,
$$

a equação para h fica,

$$
\begin{aligned}
h(\rho,\varphi) &= u_0(\rho,\varphi) \\
&\quad + \sum_{j\mu} u_\mu(\lambda_{j\mu}\rho) \frac{1}{Y_\mu(\lambda_{j\mu} a)} [b_{1\mu} \cos\mu\varphi + b_{2\mu} sen\, \mu\varphi]\,.
\end{aligned}
$$

Temos uma série de Fourier,

$$
\begin{aligned}
& f(\varphi) = \frac{A_0}{2} + \sum_{j=1}^{\infty} (A_j \cos j\varphi + B_j sen\, j\varphi)\,, \\
& A_j = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos jx\, dx\,, \\
& B_j = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) sen\, jx\, dx\,, \quad j = 0,1,2,\ldots
\end{aligned}
$$

logo,

$$
\begin{aligned}
\frac{A_0(\rho)}{2} &= \sum_j u_0(\lambda_{j0}\rho) \frac{1}{Y_0(\lambda_{j0} a)} b_{10} = \frac{1}{2\pi} \int_{-\pi}^{\pi} [h(\rho,x) - u_0(\rho,x)] dx\,, \\
A_\mu(\rho) &= \sum_j u_\mu(\lambda_{j\mu}\rho) \frac{1}{Y_\mu(\lambda_{j\mu} a)} b_{1\mu} = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos\mu x\, [h(\rho,x) - u_0(\rho,x)] dx\,, \\
B_\mu(\rho) &= \sum_j u_\mu(\lambda_{j\mu}\rho) \frac{1}{Y_\mu(\lambda_{j\mu} a)} b_{2\mu} = \frac{1}{\pi} \int_{-\pi}^{\pi} sen\, \mu x\, [h(\rho,x) - u_0(\rho,x)] dx\,.
\end{aligned}
$$

Agora temos séries de funções de Bessel,

$$f(\rho) = \sum_{j=1} A_j u_\mu(\lambda_{j\mu}\rho)\,,$$

$$A_j = \frac{\int_a^b \rho f(\rho) u_\mu(\lambda_{j\mu}\rho) d\rho}{\int_a^b \rho [u_\mu(\lambda_{j\mu}\rho)]^2 d\rho}\,,$$

assim,

$$\frac{1}{Y_0(\lambda_{j0}a)} b_{10} = \frac{1}{\int_a^b \rho [u_0(\lambda_{j0}\rho)]^2 d\rho} \int_a^b \rho \frac{A_0(\rho)}{2} u_0(\lambda_{j0}\rho) d\rho$$

$$\frac{1}{Y_\mu(\lambda_{j\mu}a)} b_{1\mu} = \frac{1}{\int_a^b \rho [u_\mu(\lambda_{j\mu}\rho)]^2 d\rho} \int_a^b \rho A_\mu(\rho) u_\mu(\lambda_{j\mu}\rho) d\rho$$

$$\frac{1}{Y_\mu(\lambda_{j\mu}a)} b_{2\mu} = \frac{1}{\int_a^b \rho [u_\mu(\lambda_{j\mu}\rho)]^2 d\rho} \int_a^b \rho B_\mu(\rho) u_\mu(\lambda_{j\mu}\rho) d\rho\,,$$

ou,

$$b_{10} = \frac{Y_0(\lambda_{j0}a)}{\int_a^b \rho [u_0(\lambda_{j0}\rho)]^2 d\rho} \int_a^b \rho u_0(\lambda_{j0}\rho) d\rho \times$$
$$\times \frac{1}{2\pi} \int_{-\pi}^{\pi} [h(\rho, x) - u_0(\rho, x)] dx\,,$$
$$b_{1\mu} = \frac{Y_\mu(\lambda_{j\mu}a)}{\int_a^b \rho [u_\mu(\lambda_{j\mu}\rho)]^2 d\rho} \int_a^b \rho u_\mu(\lambda_{j\mu}\rho) d\rho \times$$
$$\times \frac{1}{\pi} \int_{-\pi}^{\pi} \cos \mu x\, [h(\rho, x) - u_0(\rho, x)] dx\,,$$
$$b_{2\mu} = \frac{Y_\mu(\lambda_{j\mu}a)}{\int_a^b \rho [u_\mu(\lambda_{j\mu}\rho)]^2 d\rho} \int_a^b \rho u_\mu(\lambda_{j\mu}\rho) d\rho \times$$
$$\times \frac{1}{\pi} \int_{-\pi}^{\pi} sen\, \mu x\, [h(\rho, x) - u_0(\rho, x)] dx\,.$$

A solução é portanto,

$$
\begin{aligned}
u(t,\rho,\varphi) &= u_0(\rho,\varphi) \\
&+\sum_{j0}\frac{u_0(\lambda_j\rho)}{Y_0(\lambda_j a)}\cos(\lambda_j vt)\times \\
&\times\frac{Y_0(\lambda_{j0}a)}{\int_a^b\rho[u_0(\lambda_{j0}\rho)]^2d\rho}\int_a^b\rho u_0(\lambda_{j0}\rho)d\rho\times \\
&\times\frac{1}{2\pi}\int_{-\pi}^{\pi}[h(\rho,x)-u_0(\rho,x)]dx\,, \\
&+\sum_{j\mu}\frac{u_\mu(\lambda_j\rho)}{Y_\mu(\lambda_j a)}\cos(\lambda_j vt)\cos\mu\varphi\times \\
&\times\frac{Y_\mu(\lambda_{j\mu}a)}{\int_a^b\rho[u_\mu(\lambda_{j\mu}\rho)]^2d\rho}\int_a^b\rho u_\mu(\lambda_{j\mu}\rho)d\rho\times \\
&\times\frac{1}{\pi}\int_{-\pi}^{\pi}\cos\mu x\,[h(\rho,x)-u_0(\rho,x)]dx \\
&+\sum_{j\mu}\frac{u_\mu(\lambda_j\rho)}{Y_\mu(\lambda_j a)}\cos(\lambda_j vt)sen\,\mu\varphi\times \\
&\times\frac{Y_\mu(\lambda_{j\mu}a)}{\int_a^b\rho[u_\mu(\lambda_{j\mu}\rho)]^2d\rho}\int_a^b\rho u_\mu(\lambda_{j\mu}\rho)d\rho\times \\
&\times\frac{1}{\pi}\int_{-\pi}^{\pi}sen\,\mu x\,[h(\rho,x)-u_0(\rho,x)]dx\,.
\end{aligned}
$$

13. Calcule $u(t,\rho,\varphi,z)$ em $0\le\rho\le a$, $0\le z\le L$ com as condições de contorno $u(t,a,\varphi,z)=f(\varphi,z)$, $u(t,\rho,\varphi,0)=g(\rho,\varphi)$, $u(t,\rho,\varphi,L)=h(\rho,\varphi)$, $u(0,\rho,\varphi,z)=w(\rho,\varphi,z)$.

Escrevemos a solução como,

$$
\begin{aligned}
u(t,\rho,\varphi,z) &= c_0+d_0u_0(\rho,\varphi,z) \\
&+\sum_{\beta\alpha\mu}J_\mu(\beta\rho)\times \\
&\times[d_{1\alpha}\cos\alpha z+d_{2\alpha}sen\,\alpha z]\times \\
&\times[b_{1\mu}\cos\mu\varphi+b_{2\mu}sen\,\mu\varphi]\times \\
&\times[a_1\cos(\lambda vt)+a_2 sen\,(\lambda vt)]\,, \\
&\lambda^2-\alpha^2=\beta^2\,,\ \ \lambda^2>\alpha^2\,,
\end{aligned}
$$

e as condições de contorno,

$$
\begin{aligned}
u(t,a,\varphi,z) &= f(\varphi,z) = c_0 + d_0 u_0(a,\varphi,z) \\
&+ \sum_{\beta\alpha\mu} J_\mu(\beta a) \times \\
&\times [d_{1\alpha}\cos\alpha z + d_{2\alpha} sen\, \alpha z] \times \\
&\times [b_{1\mu}\cos\mu\varphi + b_{2\mu} sen\, \mu\varphi] \times \\
&\times [a_1 \cos(\lambda v t) + a_2 sen\,(\lambda v t)]\,, \\
u(t,\rho,\varphi,0) &= g(\rho,\varphi) = c_0 + d_0 u_0(\rho,\varphi,0) \\
&+ \sum_{\beta\alpha\mu} J_\mu(\beta \rho) \times \\
&\times d_{1\alpha} \times \\
&\times [b_{1\mu}\cos\mu\varphi + b_{2\mu} sen\, \mu\varphi] \times \\
&\times [a_1 \cos(\lambda v t) + a_2 sen\,(\lambda v t)]\,, \\
u(t,\rho,\varphi,L) &= h(\rho,\varphi) = c_0 + d_0 u_0(\rho,\varphi,L) \\
&+ \sum_{\beta\alpha\mu} J_\mu(\beta \rho) \times \\
&\times [d_{1\alpha}\cos\alpha L + d_{2\alpha} sen\, \alpha L] \times \\
&\times [b_{1\mu}\cos\mu\varphi + b_{2\mu} sen\, \mu\varphi] \times \\
&\times [a_1 \cos(\lambda v t) + a_2 sen\,(\lambda v t)]\,, \\
u(0,\rho,\varphi,z) &= w(\rho,\varphi,z) = c_0 + d_0 u_0(\rho,\varphi,z) \\
&+ \sum_{\beta\alpha\mu} J_\mu(\beta \rho) \times \\
&\times [d_{1\alpha}\cos\alpha z + d_{2\alpha} sen\, \alpha z] \times \\
&\times [b_{1\mu}\cos\mu\varphi + b_{2\mu} sen\, \mu\varphi] \times \\
&\times a_1\,, \\
&\lambda^2 - \alpha^2 = \beta^2\,, \quad \lambda^2 > \alpha^2\,.
\end{aligned}
$$

Para satisfazer as condições acima escolhemos,

$$
\begin{aligned}
&c_0 = 0\,, \quad d_0 = 1\,, \\
&J_\mu(\beta_{j\mu} a) = 0\,, \quad j = 1, 2, \ldots \\
&d_{1\alpha} = 0\,, \quad d_{2\alpha} = 1\,, \\
&a_1 = 1\,, \quad a_2 = 0\,, \\
&\alpha_k L = k\pi\,,
\end{aligned}
$$

logo,

$$\begin{aligned}
f(\varphi,z) &= u_0(a,\varphi,z)\\
g(\rho,\varphi) &= u_0(\rho,\varphi,0)\\
h(\rho,\varphi) &= u_0(\rho,\varphi,L)\\
w(\rho,\varphi,z) &= u_0(\rho,\varphi,z)\\
&\quad + \sum_{jk\mu} J_\mu(\beta_{j\mu}\rho) sen\,\alpha_k z \times\\
&\quad \times [b_{1\mu}\cos\mu\varphi + b_{2\mu} sen\,\mu\varphi]\,,\\
&\quad \lambda^2-\alpha^2=\beta^2\,,\ \ \lambda^2>\alpha^2\,.
\end{aligned}$$

A equação para w é uma série de Fourier, uma série de Fourier de senos, e uma série de funções de Bessel. Da expressão de uma série de Fourier,

$$\begin{aligned}
&f(\varphi)=\frac{A_0}{2}+\sum_{j=1}^{\infty}(A_j\cos j\varphi + B_j sen\,j\varphi)\,,\\
&A_j=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos jx\,dx\,,\\
&B_j=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x) sen\,jx\,dx\,,\ \ j=0,1,2,\ldots
\end{aligned}$$

temos,

$$\begin{aligned}
\frac{A_0}{2} &= \sum_{jk} b_{10}J_0(\beta_{j0}\rho) sen\,\alpha_k z = \frac{1}{2\pi}\int_{-\pi}^{\pi}[w(\rho,x,z)-u_0(\rho,x,z)]\,dx\\
A_\mu &= \sum_{jk} b_{1\mu}J_\mu(\beta_{j\mu}\rho) sen\,\alpha_k z = \frac{1}{\pi}\int_{-\pi}^{\pi}[w(\rho,x,z)-u_0(\rho,x,z)]\cos\mu x\,dx\\
B_\mu &= \sum_{jk} b_{2\mu}J_\mu(\beta_{j\mu}\rho) sen\,\alpha_k z = \frac{1}{\pi}\int_{-\pi}^{\pi}[w(\rho,x,z)-u_0(\rho,x,z)] sen\,\mu x\,dx\,.
\end{aligned}$$

As expressões acima são séries de Fourier de senos,

$$\begin{aligned}
&f(z)=\sum_{j=1}^{\infty} B_j sen\,(j\pi z/L)\,,\\
&A_j=0\,,\\
&B_j=\frac{2}{L}\int_0^L f(x) sen\,(j\pi x/L)\,dx\,,\ \ j=0,1,2,\ldots
\end{aligned}$$

logo,

$$
\begin{aligned}
\sum_j b_{10} J_0(\beta_{j0}\rho) &= \frac{2}{L}\int_0^L sen\,(k\pi z/L)\,dz \times \\
&\times \frac{1}{2\pi}\int_{-\pi}^{\pi} [w(\rho,x,z) - u_0(\rho,x,z)]\,dx\,, \\
\sum_j b_{1\mu} J_\mu(\beta_{j\mu}\rho) &= \frac{2}{L}\int_0^L sen\,(k\pi z/L)\,dz \times \\
&\times \frac{1}{\pi}\int_{-\pi}^{\pi} [w(\rho,x,z) - u_0(\rho,x,z)]\cos\mu x\,dx\,, \\
\sum_j b_{2\mu} J_\mu(\beta_{j\mu}\rho) &= \frac{2}{L}\int_0^L sen\,(k\pi z/L)\,dz \times \\
&\times \frac{1}{\pi}\int_{-\pi}^{\pi} [w(\rho,x,z) - u_0(\rho,x,z)] sen\,\mu x\,dx\,.
\end{aligned}
$$

Temos agora séries de funções de Bessel,

$$
f(\rho) = \sum_{j=1} A_j J_n(\beta_{jn}\rho)\,,
$$

$$
A_j = \frac{2}{a^2 J_{n+1}^2(\beta_{jn}a)}\int_0^a \rho J_n(\beta_{jn}\rho) f(\rho) d\rho\,,
$$

assim,

$$
\begin{aligned}
b_{10} &= \frac{2}{a^2 J_1^2(\beta_{j0}a)}\int_0^a \rho J_0(\beta_{j0}\rho) d\rho \frac{2}{L}\int_0^L sen\,(k\pi z/L)\,dz \times \\
&\times \frac{1}{2\pi}\int_{-\pi}^{\pi} [w(\rho,x,z) - u_0(\rho,x,z)]\,dx\,, \\
b_{1\mu} &= \frac{2}{a^2 J_{\mu+1}^2(\beta_{j\mu}a)}\int_0^a \rho J_\mu(\beta_{j\mu}\rho) d\rho \frac{2}{L}\int_0^L sen\,(k\pi z/L)\,dz \times \\
&\times \frac{1}{\pi}\int_{-\pi}^{\pi} [w(\rho,x,z) - u_0(\rho,x,z)]\cos\mu x\,dx\,, \\
b_{2\mu} &= \frac{2}{a^2 J_{\mu+1}^2(\beta_{j\mu}a)}\int_0^a \rho J_\mu(\beta_{j\mu}\rho) d\rho \frac{2}{L}\int_0^L sen\,(k\pi z/L)\,dz \times \\
&\times \frac{1}{\pi}\int_{-\pi}^{\pi} [w(\rho,x,z) - u_0(\rho,x,z)] sen\,\mu x\,dx\,.
\end{aligned}
$$

A solução é portanto,

$$
\begin{aligned}
u(t,\rho,\varphi,z) &= u_0(\rho,\varphi,z) \\
&+\sum_{jk} J_0(\beta_{j0}\rho) sen\,\alpha_k z \cos(\lambda v t)\times \\
&\times\frac{2}{a^2J_1^2(\beta_{j0}a)}\int_0^a \rho J_0(\beta_{j0}\rho)d\rho\frac{2}{L}\int_0^L sen\,(k\pi z/L)\,dz\times \\
&\times\frac{1}{2\pi}\int_{-\pi}^{\pi}[w(\rho,x,z)-u_0(\rho,x,z)]\,dx\,, \\
&+\sum_{jk\mu} J_\mu(\beta_{j\mu}\rho) sen\,\alpha_k z \cos\mu\varphi\cos(\lambda v t)\times \\
&\times\frac{2}{a^2J_{\mu+1}^2(\beta_{j\mu}a)}\int_0^a \rho J_\mu(\beta_{j\mu}\rho)d\rho\frac{2}{L}\int_0^L sen\,(k\pi z/L)\,dz\times \\
&\times\frac{1}{\pi}\int_{-\pi}^{\pi}[w(\rho,x,z)-u_0(\rho,x,z)]\cos\mu x\,dx\,, \\
&+\sum_{jk\mu} J_\mu(\beta_{j\mu}\rho) sen\,\alpha_k z sen\,\mu\varphi\cos(\lambda v t)\times \\
&\times\frac{2}{a^2J_{\mu+1}^2(\beta_{j\mu}a)}\int_0^a \rho J_\mu(\beta_{j\mu}\rho)d\rho\frac{2}{L}\int_0^L sen\,(k\pi z/L)\,dz\times \\
&\times\frac{1}{\pi}\int_{-\pi}^{\pi}[w(\rho,x,z)-u_0(\rho,x,z)] sen\,\mu x\,dx\,, \\
&\lambda^2-\alpha^2=\beta^2\,,\quad \lambda^2>\alpha^2\,.
\end{aligned}
$$

14. Calcule $u(t,\rho,\varphi,z)$ em $0\le\rho\le a$, $0\le z\le L$ com as condições de contorno,

$$
\begin{aligned}
&u(t,a,\varphi,z)=\mu(t,\varphi,z)\,,\\
&u(t,\rho,\varphi,0)=\nu_1(t,\rho,\varphi)\,,\\
&u(t,\rho,\varphi,L)=\nu_2(t,\rho,\varphi)\,,\\
&u(0,\rho,\varphi,z)=\varphi(\rho,\varphi,z)\,,\\
&u_t(0,\rho,\varphi,z)=\psi(\rho,\varphi,z)\,.
\end{aligned}
$$

15. Calcule $u(t,\rho,\varphi,z)$ em $a\le\rho\le b$, $0\le z\le L$ com as condições de contorno $u(t,a,\varphi,z)=f(\varphi,z)$, $u(t,b,\varphi,z)=g(\varphi,z)$, $u(t,\rho,\varphi,0)=h(\rho,\varphi)$, $u(t,\rho,\varphi,L)=w(\rho,\varphi)$, $u(0,\rho,\varphi,z)=v(\rho,\varphi,z)$.

Escrevemos a solução como,

$$\begin{aligned}
u(t,\rho,\varphi,z) &= c_0 + d_0 u_0(\rho,\varphi,z) \\
&+ \sum_{\beta\alpha\mu} [c_1 J_\mu(\beta\rho) + c_2 Y_\mu(\beta\rho)] \times \\
&\times [d_{1\alpha} \cos \alpha z + d_{2\alpha} sen\, \alpha z] \times \\
&\times [b_{1\mu} \cos \mu\varphi + b_{2\mu} sen\, \mu\varphi] \times \\
&\times [a_1 \cos(\lambda v t) + a_2 sen\, (\lambda v t)]\,, \\
&\lambda^2 - \alpha^2 = \beta^2\,, \;\; \lambda^2 > \alpha^2\,,
\end{aligned}$$

e as condições de contorno,

$$
\begin{aligned}
u(t,a,\varphi,z) &= f(\varphi,z) = c_0 + d_0 u_0(a,\varphi,z) \\
&+ \sum_{\beta\alpha\mu} [c_1 J_\mu(\beta a) + c_2 Y_\mu(\beta a)] \times \\
&\times [d_{1\alpha} \cos\alpha z + d_{2\alpha} sen\,\alpha z] \times \\
&\times [b_{1\mu} \cos\mu\varphi + b_{2\mu} sen\,\mu\varphi] \times \\
&\times [a_1 \cos(\lambda v t) + a_2 sen\,(\lambda v t)]\,, \\
u(t,b,\varphi,z) &= g(\varphi,z) = c_0 + d_0 u_0(b,\varphi,z) \\
&+ \sum_{\beta\alpha\mu} [c_1 J_\mu(\beta b) + c_2 Y_\mu(\beta b)] \times \\
&\times [d_{1\alpha} \cos\alpha z + d_{2\alpha} sen\,\alpha z] \times \\
&\times [b_{1\mu} \cos\mu\varphi + b_{2\mu} sen\,\mu\varphi] \times \\
&\times [a_1 \cos(\lambda v t) + a_2 sen\,(\lambda v t)]\,, \\
u(t,\rho,\varphi,0) &= h(\rho,\varphi) = c_0 + d_0 u_0(\rho,\varphi,0) \\
&+ \sum_{\beta\alpha\mu} [c_1 J_\mu(\beta \rho) + c_2 Y_\mu(\beta \rho)] \times \\
&\times d_{1\alpha} \times \\
&\times [b_{1\mu} \cos\mu\varphi + b_{2\mu} sen\,\mu\varphi] \times \\
&\times [a_1 \cos(\lambda v t) + a_2 sen\,(\lambda v t)]\,, \\
u(t,\rho,\varphi,L) &= w(\rho,\varphi) = c_0 + d_0 u_0(\rho,\varphi,L) \\
&+ \sum_{\beta\alpha\mu} [c_1 J_\mu(\beta \rho) + c_2 Y_\mu(\beta \rho)] \times \\
&\times [d_{1\alpha} \cos\alpha L + d_{2\alpha} sen\,\alpha L] \times \\
&\times [b_{1\mu} \cos\mu\varphi + b_{2\mu} sen\,\mu\varphi] \times \\
&\times [a_1 \cos(\lambda v t) + a_2 sen\,(\lambda v t)]\,, \\
u(0,\rho,\varphi,z) &= v(\rho,\varphi,z) = c_0 + d_0 u_0(\rho,\varphi,z) \\
&+ \sum_{\beta\alpha\mu} [c_1 J_\mu(\beta \rho) + c_2 Y_\mu(\beta \rho)] \times \\
&\times [d_{1\alpha} \cos\alpha z + d_{2\alpha} sen\,\alpha z] \times \\
&\times [b_{1\mu} \cos\mu\varphi + b_{2\mu} sen\,\mu\varphi] \times \\
&\times a_1\,, \\
&\lambda^2 - \alpha^2 = \beta^2\,, \quad \lambda^2 > \alpha^2\,,
\end{aligned}
$$

Satisfazemos as condições acima escolhendo,

$$
\begin{aligned}
&c_0 = 0\,, \quad d_0 = 1\,,\\
&c_1 J_\mu(\beta a) + c_2 Y_\mu(\beta a) = 0\,,\\
&c_1 J_\mu(\beta b) + c_2 Y_\mu(\beta b) = 0\,,\\
&d_{1\alpha} = 0\,, \quad d_{2\alpha} = 1\,,\\
&\alpha_k L = k\pi\,,\\
&a_1 = 1\,, \quad a_2 = 0\,,
\end{aligned}
$$

logo,

$$
\begin{aligned}
f(\varphi, z) &= u_0(a, \varphi, z)\,,\\
g(\varphi, z) &= u_0(b, \varphi, z)\,,\\
h(\rho, \varphi) &= u_0(\rho, \varphi, 0)\,,\\
w(\rho, \varphi) &= u_0(\rho, \varphi, L)\,,\\
v(\rho, \varphi, z) &= u_0(\rho, \varphi, z)\\
&\quad + \sum_{\beta\alpha\mu} [c_1 J_\mu(\beta\rho) + c_2 Y_\mu(\beta\rho)] \times\\
&\quad \times [d_{1\alpha} \cos \alpha z + d_{2\alpha} sen\, \alpha z] \times\\
&\quad \times [b_{1\mu} \cos \mu\varphi + b_{2\mu} sen\, \mu\varphi]\,,\\
&\quad \lambda^2 - \alpha^2 = \beta^2\,, \quad \lambda^2 > \alpha^2\,.
\end{aligned}
$$

Podemos expressar c_2 *como,*

$$
c_2 = -c_1 \frac{J_\mu(\beta a)}{Y_\mu(\beta a)}\,,
$$

Também temos,

$$
J_\mu(\beta_{j\mu} a) Y_\mu(\beta_{j\mu} b) - J_\mu(\beta_{j\mu} b) Y_\mu(\beta_{j\mu} a) = 0\,, \quad j = 1, 2, \ldots
$$

Assim a equação para v *fica, fazendo* $c_1 = 1$*,*

$$
\begin{aligned}
v(\rho, \varphi, z) &= u_0(\rho, \varphi, z)\\
&\quad + \sum_{jk\mu} [J_\mu(\beta_{j\mu}\rho) Y_\mu(\beta_{j\mu} a) - J_\mu(\beta_{j\mu} a) Y_\mu(\beta_{j\mu}\rho)] \frac{1}{Y_\mu(\beta_{j\mu} a)} \times\\
&\quad \times sen\, \alpha_k z \times\\
&\quad \times [b_{1\mu} \cos \mu\varphi + b_{2\mu} sen\, \mu\varphi]\,,
\end{aligned}
$$

ou,

$$\begin{aligned} v(\rho,\varphi,z) &= u_0(\rho,\varphi,z) \\ &+\sum_{jk\mu} u_\mu(\beta_{j\mu}\rho)\frac{1}{Y_\mu(\beta_{j\mu}a)}sen\,\alpha_k z\times \\ &\times[b_{1\mu}\cos\mu\varphi + b_{2\mu}sen\,\mu\varphi]\,. \end{aligned}$$

Temos uma série de Fourier,

$$\begin{aligned} & f(\varphi)=\frac{A_0}{2}+\sum_{j=1}^{\infty}(A_j\cos j\varphi + B_j sen\,j\varphi)\,, \\ & A_j=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos jx\,dx\,, \\ & B_j=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sen\,jx\,dx\,, \quad j=0,1,2,\ldots \end{aligned}$$

logo,

$$\begin{aligned} \frac{A_0}{2} &= \sum_{jk} u_0(\beta_{j0}\rho)\frac{1}{Y_0(\beta_{j0}a)}sen\,\alpha_k z b_{10}\,, \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}[v(\rho,x,z)-u_0(\rho,x,z)]\,dx\,, \\ A_\mu &= \sum_{jk} u_\mu(\beta_{j\mu}\rho)\frac{1}{Y_\mu(\beta_{j\mu}a)}sen\,\alpha_k z b_{1\mu}\,, \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi}\cos\mu x[v(\rho,x,z)-u_0(\rho,x,z)]\,dx\,, \\ B_\mu &= \sum_{jk} u_\mu(\beta_{j\mu}\rho)\frac{1}{Y_\mu(\beta_{j\mu}a)}sen\,\alpha_k z b_{2\mu}\,, \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi}sen\,\mu x[v(\rho,x,z)-u_0(\rho,x,z)]\,dx\,. \end{aligned}$$

Uma série de Fourier de senos é,

$$
\begin{aligned}
&f(z) = \sum_{j=1}^{\infty} B_j sen\,(j\pi z/L)\,,\\
&A_j = 0\,,\\
&B_j = \frac{2}{L}\int_0^L f(x) sen\,(j\pi x/L)\,dx\,,\quad j = 0, 1, 2, \ldots
\end{aligned}
$$

assim,

$$
\begin{aligned}
\sum_j u_0(\beta_{j0}\rho)\frac{1}{Y_0(\beta_{j0}a)} b_{10} &= \frac{2}{L}\int_0^L sen\,(k\pi z/L)\,dz \frac{A_0(\rho,z)}{2}\,,\\
&= \frac{2}{L}\int_0^L sen\,(k\pi z/L)\,dz \times\\
&\quad \times \frac{1}{2\pi}\int_{-\pi}^{\pi} [v(\rho,x,z) - u_0(\rho,x,z)]\,dx\,,\\
\sum_j u_\mu(\beta_{j\mu}\rho)\frac{1}{Y_\mu(\beta_{j\mu}a)} b_{1\mu} &= \frac{2}{L}\int_0^L sen\,(k\pi z/L)\,dz A_\mu(\rho,z)\,,\\
&= \frac{2}{L}\int_0^L sen\,(k\pi z/L)\,dz \times\\
&\quad \times \frac{1}{\pi}\int_{-\pi}^{\pi} \cos\mu x [v(\rho,x,z) - u_0(\rho,x,z)]\,dx\,,\\
\sum_j u_\mu(\beta_{j\mu}\rho)\frac{1}{Y_\mu(\beta_{j\mu}a)} b_{2\mu} &= \frac{2}{L}\int_0^L sen\,(k\pi z/L)\,dz B_\mu(\rho,z)\,,\\
&= \frac{2}{L}\int_0^L sen\,(k\pi z/L)\,dz \times\\
&\quad \times \frac{1}{\pi}\int_{-\pi}^{\pi} sen\,\mu x [v(\rho,x,z) - u_0(\rho,x,z)]\,dx\,.
\end{aligned}
$$

Temos, por fim, séries em funções de Bessel,

$$
\begin{aligned}
&f(\rho) = \sum_{j=1} A_j u_\mu(\beta_{j\mu}\rho)\,,\\
&A_j = \frac{1}{\int_a^b \rho[u_\mu(\beta_{j\mu}\rho)]^2 d\rho}\int_a^b \rho f(\rho) u_\mu(\beta_{j\mu}\rho) d\rho\,,
\end{aligned}
$$

o que nos dá,

$$
\begin{aligned}
b_{10} &= \frac{Y_0(\beta_{j0}a)}{\int_a^b \rho[u_0(\beta_{j0}\rho)]^2 d\rho}\int_a^b \rho u_0(\beta_{j0}\rho)d\rho\frac{2}{L}\int_0^L sen\,(k\pi z/L)\,dz \times \\
&\quad \times\frac{1}{2\pi}\int_{-\pi}^{\pi}[v(\rho,x,z)-u_0(\rho,x,z)]\,dx \\
b_{1\mu} &= \frac{Y_\mu(\beta_{j\mu}a)}{\int_a^b \rho[u_\mu(\beta_{j\mu}\rho)]^2 d\rho}\int_a^b \rho u_\mu(\beta_{j\mu}\rho)d\rho\frac{2}{L}\int_0^L sen\,(k\pi z/L)\,dz \times \\
&\quad \times\frac{1}{\pi}\int_{-\pi}^{\pi}\cos\mu x[v(\rho,x,z)-u_0(\rho,x,z)]\,dx\,, \\
b_{2\mu} &= \frac{Y_\mu(\beta_{j\mu}a)}{\int_a^b \rho[u_\mu(\beta_{j\mu}\rho)]^2 d\rho}\int_a^b \rho u_\mu(\beta_{j\mu}\rho)d\rho\frac{2}{L}\int_0^L sen\,(k\pi z/L)\,dz \times \\
&\quad \times\frac{1}{\pi}\int_{-\pi}^{\pi} sen\,\mu x[v(\rho,x,z)-u_0(\rho,x,z)]\,dx\,.
\end{aligned}
$$

A solução é portanto,

$$
\begin{aligned}
u(t,\rho,\varphi,z) \;=\;& u_0(\rho,\varphi,z) \\
&+\sum_{jk}\frac{u_0(\beta_{j0}\rho)}{Y_0(\beta_{j0}a)} sen\,\alpha_k z\cos(\lambda vt)\,\times \\
&\times\frac{Y_0(\beta_{j0}a)}{\int_a^b\rho[u_0(\beta_{j0}\rho)]^2d\rho}\int_a^b\rho u_0(\beta_{j0}\rho)d\rho\frac{2}{L}\int_0^L sen\,(k\pi z/L)\,dz\,\times \\
&\times\frac{1}{2\pi}\int_{-\pi}^{\pi}[v(\rho,x,z)-u_0(\rho,x,z)]\,dx\,, \\
&+\sum_{jk\mu}\frac{u_\mu(\beta_{j\mu}\rho)}{Y_\mu(\beta_{j\mu}a)} sen\,\alpha_k z\cos\mu\varphi\cos(\lambda vt)\,\times \\
&\times\frac{Y_\mu(\beta_{j\mu}a)}{\int_a^b\rho[u_\mu(\beta_{j\mu}\rho)]^2d\rho}\int_a^b\rho u_\mu(\beta_{j\mu}\rho)d\rho\frac{2}{L}\int_0^L sen\,(k\pi z/L)\,dz\,\times \\
&\times\frac{1}{\pi}\int_{-\pi}^{\pi}\cos\mu x[v(\rho,x,z)-u_0(\rho,x,z)]\,dx\,, \\
&+\sum_{jk\mu}\frac{u_\mu(\beta_{j\mu}\rho)}{Y_\mu(\beta_{j\mu}a)} sen\,\alpha_k z sen\,\mu\varphi\cos(\lambda vt)\,\times \\
&\times\frac{Y_\mu(\beta_{j\mu}a)}{\int_a^b\rho[u_\mu(\beta_{j\mu}\rho)]^2d\rho}\int_a^b\rho u_\mu(\beta_{j\mu}\rho)d\rho\frac{2}{L}\int_0^L sen\,(k\pi z/L)\,dz\,\times \\
&\times\frac{1}{\pi}\int_{-\pi}^{\pi} sen\,\mu x[v(\rho,x,z)-u_0(\rho,x,z)]\,dx\,, \\
&\lambda^2-\alpha^2=\beta^2\,,\quad \lambda^2>\alpha^2\,.
\end{aligned}
$$

16. Calcule $u(t,\rho,\varphi,z)$ em $a\le\rho\le b$, $0\le z\le L$ com as condições de contorno,

$$
\begin{aligned}
&u(t,a,\varphi,z)=\mu_1(t,\varphi,z)\,,\\
&u(t,b,\varphi,z)=\mu_2(t,\varphi,z)\,,\\
&u(t,\rho,\varphi,0)=\nu_1(t,\rho,\varphi)\,,\\
&u(t,\rho,\varphi,L)=\nu_2(t,\rho,\varphi)\,,\\
&u(0,\rho,\varphi,z)=\varphi(\rho,\varphi,z)\,,\\
&u_t(0,\rho,\varphi,z)=\psi(\rho,\varphi,z)\,.
\end{aligned}
$$

17. Calcule $u(t,\rho,\varphi,z)$ para,

(a)

$$\frac{\partial^2 u}{\partial t^2} = v^2 \nabla^2 u + F(\rho, \varphi, z)\,,$$

(b)

$$\frac{\partial^2 u}{\partial t^2} = v^2 \nabla^2 u + \alpha^2\,,$$

com α constante,

(c)

$$\frac{\partial^2 u}{\partial t^2} = v^2 \nabla^2 u + \alpha^2 u\,,$$

com α constante e,

$$\begin{aligned}
&0 \le \rho \le a\,, \quad 0 \le \varphi \le 2\pi\,, \quad 0 \le z \le L\,,\\
&u(t, a, \varphi, z) = \mu_1(\varphi, z)\,,\\
&u(t, \rho, \varphi, 0) = \nu_1(\rho, \varphi)\,, \quad u(\rho, \varphi, L) = \nu_2(\rho, \varphi)\,,\\
&u(0, \rho, \varphi, z) = \varphi(\rho, \varphi, z)\,, \quad u_t(0, \rho, \varphi, z) = \psi(\rho, \varphi, z)\,.
\end{aligned}$$

18. Calcule $u(t, \rho, \varphi, z)$ para,

(a)

$$\frac{\partial^2 u}{\partial t^2} = v^2 \nabla^2 u + F(t, \rho, \varphi, z)\,,$$

(b)

$$\frac{\partial^2 u}{\partial t^2} = v^2 \nabla^2 u + \alpha^2\,,$$

com α constante,

(c)

$$\frac{\partial^2 u}{\partial t^2} = v^2 \nabla^2 u + \alpha^2 u\,,$$

com α constante e,

$$\begin{aligned}
&0 \le \rho \le a\,, \quad 0 \le \varphi \le 2\pi\,, \quad 0 \le z \le L\,,\\
&u(t, a, \varphi, z) = \mu_1(t, \varphi, z)\,,\\
&u(t, \rho, \varphi, 0) = \nu_1(t, \rho, \varphi)\,, \quad u(t, \rho, \varphi, L) = \nu_2(t, \rho, \varphi)\,,\\
&u(0, \rho, \varphi, z) = \varphi(\rho, \varphi, z)\,, \quad u_t(0, \rho, \varphi, z) = \psi(\rho, \varphi, z)\,.
\end{aligned}$$

19. Calcule $u(t, \rho, \varphi, z)$ para,

(a)

$$\frac{\partial^2 u}{\partial t^2} = v^2 \nabla^2 u + F(\rho, \varphi, z)\,,$$

(b)

$$\frac{\partial^2 u}{\partial t^2} = v^2 \nabla^2 u + \alpha^2 \,,$$

com α constante,

(c)

$$\frac{\partial^2 u}{\partial t^2} = v^2 \nabla^2 u + \alpha^2 u \,,$$

com α constante e,

$$\begin{aligned}
& a \leq \rho \leq b \,, \quad 0 \leq \varphi \leq 2\pi \,, \quad 0 \leq z \leq L \,, \\
& u(t,a,\varphi,z) = \mu_1(\varphi,z) \,, \quad u(t,b,\varphi,z) = \mu_2(\varphi,z) \,, \\
& u(t,\rho,\varphi,0) = \nu_1(\rho,\varphi) \,, \quad u(t,\rho,\varphi,L) = \nu_2(\rho,\varphi) \,, \\
& u(0,\rho,\varphi,z) = \varphi(\rho,\varphi,z) \,, \quad u_t(0,\rho,\varphi,z) = \psi(\rho,\varphi,z) \,.
\end{aligned}$$

20. Calcule $u(t,\rho,\varphi,z)$ para,

(a)

$$\frac{\partial^2 u}{\partial t^2} = v^2 \nabla^2 u + F(t,\rho,\varphi,z) \,,$$

(b)

$$\frac{\partial^2 u}{\partial t^2} = v^2 \nabla^2 u + \alpha^2 \,,$$

com α constante,

(c)

$$\frac{\partial^2 u}{\partial t^2} = v^2 \nabla^2 u + \alpha^2 u \,,$$

com α constante e,

$$\begin{aligned}
& a \leq \rho \leq b \,, \quad 0 \leq \varphi \leq 2\pi \,, \quad 0 \leq z \leq L \,, \\
& u(t,a,\varphi,z) = \mu_1(t,\varphi,z) \,, \quad u(t,b,\varphi,z) = \mu_2(t,\varphi,z) \,, \\
& u(t,\rho,\varphi,0) = \nu_1(t,\rho,\varphi) \,, \quad u(t,\rho,\varphi,L) = \nu_2(t,\rho,\varphi) \,, \\
& u(0,\rho,\varphi,z) = \varphi(\rho,\varphi,z) \,, \quad u_t(0,\rho,\varphi,z) = \psi(\rho,\varphi,z) \,.
\end{aligned}$$

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